4/29/13 Lecture outline

⋆ Reading: Zwiebach chapters 5, 6

• Last time: string action: $S=\int dtdx\mathcal{L}(\frac{\partial y}{\partial t},\frac{\partial y}{\partial x})$, with momentum densities

$$\mathcal{P}^t=\frac{\partial \mathcal{L}}{\partial \dot{y}}, \qquad \mathcal{P}^x=\frac{\partial \mathcal{L}}{\partial y'}.$$

Least action gives the equations of motion

$$\frac{\partial \mathcal{P}^t}{\partial t}+\frac{\partial \mathcal{P}^x}{\partial x}=0.$$

The non-relativistic string has $\mathcal{L}=\frac{1}{2}\mu_0\dot{y}^2-\frac{1}{2}T_0y'^2$, which we're going to replace with a relativistic version. For guidance, noted that a relativistic point particle of mass m has $S=-mc\int ds=-mc^2\int dt\sqrt{1-v^2/c^2}$ and noted its reparametrization invariance: write $x_\mu(\tau)$, and can change worldline parameter τ to an arbitrary new parameterization $\tau'(\tau)$, and the action is invariant. To see this use $S=-mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ and change $\frac{dx^\mu}{d\tau}=\frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$ and note that $S\to S$. Euler Lagrange equations of motion: $\frac{dp_\mu}{d\tau}=0$.

• When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action

$$S=\int(-mcds+\frac{q}{c}A_\mu dx^\mu), \tag{1}$$

which is manifestly relativistically invariant (and also repparameterization) invariant. Note also that, under a gauge transformation, we have $S\to S+\frac{qf}{c}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L=-mc\sqrt{1-\vec{v}^2/c^2}+\frac{q}{c}\vec{v}\cdot\vec{A}-q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P}=\partial L/\partial\vec{v}=m\vec{v}/\sqrt{1-\vec{v}^2/c^2}+\frac{q}{c}\vec{A}$. The Hamiltonian is $H=\vec{v}\cdot\vec{P}-L=\sqrt{m^2c^4+c^2(\vec{P}-\frac{q}{c}\vec{A})^2}+q\phi$. The equations of motion can be written as $\frac{d^2x^\mu}{d\tau^2}=\frac{q}{mc}F_{\mu\nu}\frac{dx^\nu}{d\tau}$. In the non-relativistic limit we have $H=\frac{1}{2m}(\vec{P}-\frac{q}{c}\vec{A})^2+q\phi$, where $\vec{P}-\frac{q}{c}\vec{A}=m\vec{v}$.

• Recap: $S=-mc\int ds+\frac{q}{c}\int A_\mu dx^\mu$ for a relativistic point particle, where we can write $ds=\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\tau$, with $\dot{}\equiv\frac{d}{d\tau}$, and τ is the arbitrary worldline parameter, with reparameterization symmetry $\tau\to\tau'$. For a string world-sheet, we need two parameters, ξ^a, $a=1,2$. The string trajectory is $x:\Sigma\to M$, where Σ is the 2d world-sheet, with local coordinates ξ^a, and M is the target space, with local coordinates x^μ. The worldsheet area element is $A=\int d^2\xi\sqrt{|h|}$, where h_{ab} is the worldsheet metric, and $|h|$ is its determinant.

Suppose that the target space has metric $g_{\mu\nu}$, with space-time length e.g. $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. By writing $dx^\mu = \partial_a x^\mu d\xi^a$, we get

$$ds^2 = g_{\mu\nu} \frac{dx^\mu}{d\xi^a} \frac{dx^\nu}{d\xi^b} d\xi^a d\xi^b, \qquad \text{so} \qquad h_{ab} = g_{\mu\nu} \frac{dx^\mu}{d\xi^a} \frac{dx^\nu}{d\xi^b},$$

where this h_{ab} is called the induced metric. So the worldsheet area functional is

$$A = \int d^2\xi \sqrt{\det(g_{\mu\nu} \frac{dx^\mu}{d\xi^a} \frac{dx^\nu}{d\xi^b})}.$$