6/3/13 Lecture outline

⋆ Reading: Zwiebach chapter 12.

• Open string. Imposed constraints $(\dot{X} \pm X')^2 = 0$ (via reparam choice) to simplify things. Get EOM $\partial_\sigma \mathcal{P}^{\sigma\mu} + \partial_\tau \mathcal{P}^{\tau\mu} = 0$ with

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X^{\mu'}, \qquad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu.$$

Recall the solution with N BCs:

$$X^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^I \cos n\sigma e^{-in\tau}, \tag{1}$$

$$\dot{X}^\mu \pm X^{\mu'} = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau\pm\sigma)}.$$

• In light cone gauge take $n_\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots)$. Then $n \cdot X = X^+$ and $n \cdot p = p^+$, so our constraint gives $X^+ = \beta\alpha' p^+ \tau$ and $p^+ = 2\pi \mathcal{P}^{\tau +}/\beta$ (again, $\beta = 2$ for open strings and $\beta = 1$ for closed strings. Also note, $X'^+ = 0$ and $\dot{X}^+ = \beta\alpha' p^+$); of course, p^+ is a constant of the motion. Since the constraints give $(\dot{X} \pm X')^2 = -2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2 = 0$, we can write this as $\partial_\tau X^- \pm \partial_\sigma X^- = \frac{1}{\beta\alpha'} \frac{1}{2p^+} (\dot{X}^I \pm X^{I'})^2$, where I are the transverse directions. This leads to

$$\sqrt{2\alpha'} \alpha_n^- = \frac{1}{p^+} L_n^\perp, \qquad L_n^\perp = \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^I \alpha_m^I.$$

This means that there is no dynamics in X^-, other than the zero mode. For $n = 0$, using $\alpha_0^- = \sqrt{2\alpha'} p^-$ get $2\alpha' p^+ p^- = L_0^\perp$. Light cone gauge allows us to make $\dot{X}^+$ a constant, and to solve for the derivatives of X^- (without having to take a square root). Finally, note that the string has

$$M^2 = -p^2 = 2p^+ p^- - P^I p^I = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_n^{I*} \alpha_n^I.$$

See that all classical states have $M^2 \geq 0$.

• In light cone gauge, much as with the point particle, the independent variables are $(X^I, (\sigma), x_0^-, \mathcal{P}^{\tau I}(\sigma), p^+)$. In the H picture the capitalized ones depend (implicitly) on τ too. The commutation relations are

$$[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')] = i\eta^{IJ} \delta(\sigma - \sigma'), \qquad [x_0^-, p^+] = -i.$$

The Hamiltonian is taken to be

$$H = 2\alpha' p^+ p^- = 2\alpha' p^+ \int_0^\pi d\sigma \mathcal{P}^{\tau -} = \pi\alpha' \int_0^\pi d\sigma (\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + X^{I'} X^{I'} (2\pi\alpha')^{-2})$$

Can write $H = L_0^\perp$ since $L_0^\perp = 2\alpha' p^+ p^-$.

This H properly yields the expected time derivatives, e.g. $\dot{X}^I = 2\pi\alpha' \mathcal{P}^{\tau I}$.

The needed commutators are ensured by

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ}\delta_{n+m,0}.$$

Also, as before, we define $\alpha_0^I \equiv \sqrt{2\alpha'} p^I$. Now define $\alpha_{n>0}^\mu = \sqrt{n} a_n^\mu$ and $\alpha_{-n}^\mu = a_n^{\mu *}\sqrt{n}$ to rewrite the above as

$$[a_m^I, a^{J^\dagger}] = \delta_{m,n}\eta^{IJ}. \tag{2}$$

• The transverse light cone coordinates can be described by

$$S_{l.c.} = \int d\tau d\sigma \frac{1}{4\pi\alpha'} (\dot{X}^I \dot{X}^I - X^{I'} X^{I'}).$$

Gives correct $\mathcal{P}^{\tau I} = \partial\mathcal{L}/\partial\dot{X}^I$ and correct $H = \int d\sigma(\mathcal{P}^{\tau I}\dot{X}^I - \mathcal{L})$.

Writing $X^I(\tau, \sigma) = q^I(\tau) + 2\sqrt{\alpha'} \sum_{n=1}^\infty q_N^I(\tau) n^{-1/2} \cos n\sigma$ and plugging into the action above gives

$$S = \int d\tau [\frac{1}{4\alpha'} \dot{q}^I \dot{q}^I + \sum_{n=1}^\infty (\frac{1}{2n} \dot{q}_n^I \dot{q}_n^I - \frac{n}{2} q_n^I q_n^I)]$$

and

$$H = \alpha' p^I p^I + \sum_{n=1}^\infty \frac{n}{2} (p_n^I p_n^I + q_n^I q_m^I).$$

A bunch of harmonic oscillators. Relate to (1) and (2), showing that the a_m can be interpreted as the usual harmonic oscillator annihilation operators.

• Summary: we fix X^+ to be simply related to τ, find that the X^I are given by simple harmonic oscillators, and X^- is a complicated expression, fully determined in terms of the transverse direction quantities:

$X^+(\tau\sigma) = 2\alpha' p^+ \tau = \sqrt{2\alpha'} \alpha_0^+ \tau$. For X^- recall expansion, with $\sqrt{2\alpha'} \alpha_n^- = \frac{1}{p^+} L_n^\perp$, where $L_n^\perp \equiv \frac{1}{2} \sum_p \alpha_{n-p}^I \alpha_p^I$ is the transverse Virasoro operator. Recall $[\alpha_m^I, \alpha_n^J] = m\delta^{IJ}\delta_{m+n,0}$. There is an ordering ambiguity here, only for $L_0^\perp$:

$$L_0^\perp = \tfrac{1}{2}\alpha_0\alpha_0 + \tfrac{1}{2}\sum_{p=1}^\infty \alpha_{-p}^I \alpha_p^I + \tfrac{1}{2}\sum_{p=1}^\infty \alpha_p^I \alpha_{-p}^I.$$

The ordering in the last terms need to be fixed, so the annihilation operator α_p is on the right, using $\alpha^I_p\alpha^I_{-p} = \alpha^I_{-p}\alpha^I_p + [\alpha^I_p, \alpha^I_{-p}]$, which gives

$$L_0^\perp = \alpha' p^I p^I + \sum_{n=1}^\infty n a_n^{I\dagger} a_n^I,$$

where the normal ordering constant has been put into

$$2\alpha' p^- = \frac{1}{p^+}(L_0^\perp + a), \qquad a = \tfrac{1}{2}(D-2)\sum_{p=1}^\infty p.$$

This leads to

$$M^2 = \frac{1}{\alpha'}(a + \sum_{n=1}^\infty n a_n^{I\dagger} a_n^I).$$

The divergent sum for a is regulated by using $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ and analytically continuing to get $\zeta(-1) = -1/12$. So

$$a = -\frac{1}{24}(D-2).$$

• Virasoro generators and algebra (corresponds to worldsheet energy-momentum tensor). Since $(\alpha^I_n)^\dagger = \alpha^I_{-n}$, get $L_n^{\perp\dagger} = L_{-n}^\perp$. Also,

$$[L_m^\perp, \alpha_n^I] = -n\alpha^I_{n+m}.$$

$$[L_m^\perp, L_n^\perp] = (m-n)L_{n+m}^\perp + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0}.$$

• Spacetime Lorentz symmetry corresponds to conserved currents on worldsheet, with conserved charges

$$M_{\mu\nu} = \int_0^\pi (X_\mu \mathcal{P}^\tau_\nu - (\mu \leftrightarrow \nu))d\sigma.$$

Plug in $\mathcal{P}^\tau_\nu = \frac{1}{2\alpha'}\dot{X}^\mu$ and plug in oscillator expansion of X^μ to get

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n).$$

In light cone gauge, have to be careful with M^{-I}, since X^- is constrained to something complicated,

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma, \qquad \sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp.$$

and also careful to ensure that $[M^{-I}, M^{-J}] = 0$. Find, after appropriately ordering terms,

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I(L_0^\perp + a) + (L_0^\perp + a)x_0^I\right) - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^\infty \frac{1}{n}(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp).$$

Then get

$$[M^{-I}, M^{-J}] = -\frac{1}{\alpha' p^{+2}}\sum_{m=1}^\infty (\alpha_{-m}^I \alpha_m^J - (I \leftrightarrow J))[m(1-((D-2)/24)) + m^{-1}(((D-2)/24)+a)].$$

Since this must be zero, get $D = 26$ and $a = -1$.

The worldsheet Hamiltonian is thus

$$H = 2\alpha' p^+ p^- = L_0^\perp - 1.$$