4/30/12 Lecture outline

⋆ Reading: Zwiebach chapter 7, 8, 9.

• Last time: the string generalization of $S = -mc\int ds$ is

$$S_{Nambu-Goto} = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2},$$

where we define $\dot{X}^\mu \equiv \frac{dx^\mu}{d\tau}$ and $X\mu' \equiv \frac{\partial X^\mu}{\partial \sigma}$ annd T_0 is the string tension, with $[T_0] = [F] = [ML/T^2]$.

The action is reparameterization invariant: can take $(\tau,\sigma) \to (\tau'(\tau,\sigma), \sigma'(\tau,\sigma))$ and get $S \to S$. Enormous symmetry/redundancy in choice of (τ,σ); can "fix the gauge" to some convenient choice, and the physics is completely independent of the choice.

• We can write S_{NG} in terms of the Lagrangian density

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

For the open string, $\delta S = 0$ also requires $\int d\tau [\delta X^\mu P^\sigma_\mu]_0^{\sigma_0} = 0$, which requires for each μ index either of the Dirichlet or Neumann BCs, at each end:

$$\text{Dirichlet} \qquad \frac{\partial X^\mu}{\partial \tau}(\tau,\sigma_*) = 0 \qquad \to \qquad \delta X^\mu(\tau,\sigma_*) = 0,$$

$$\text{Neumann} \qquad \mathcal{P}^\sigma_\mu(\tau,\sigma_*) = 0.$$

• Exploit $(\tau,\sigma) \to (\tau',\sigma')$ reparameterization invariance to pick useful "gauges", to simplify the above equations. We will discuss choices such that we can impose constraints

$$\dot{X}\cdot X' = 0 \qquad \dot{X}^2 + X'^2 = 0. \tag{1}$$

In this case, we have

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu'}, \tag{2}$$

and then the EOM is simply a wave equation:

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0. \tag{3}$$

Now let's explain these things in more detail.

• Static gauge: pick $\tau = t$. Verify sign inside $\sqrt{\cdot}$ in this case: $X^{\mu'} = (0, \vec{X}')$, $\dot{X}^\mu = (c, \dot{\vec{X}})$, take e.g. $\dot{\vec{X}} = 0$ to get $\sqrt{\cdot} = c|\vec{X}'|$.

• In static gauge, there is no KE, so $L = -V$, and verify that string stretched length a, e.g. $X^1 = f(\sigma)$, has $V = T_0 a$: $\dot{X}^2 \to -c^2$, $(X')^2 = (f')^2$, $\dot{X}\cdot X' = 0$, gives $V = T_0 a$. So $\mu_0 = T_0/c^2$.

• In static gauge, express S in terms of $\vec{v}_\perp = \partial_t\vec{X} - (\partial_t\vec{X}\cdot\partial_s\vec{X})\partial_s\vec{X}$ (with $ds \equiv |d\vec{X}|_{t=const} = |\partial_\sigma\vec{X}||d\sigma|$), show $(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2 = (\frac{ds}{d\sigma})^2(c^2 - v_\perp^2)$, to get $L = -T_0\int ds\sqrt{1 - v_\perp^2/c^2}$. Also get

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\frac{(\partial_s\vec{X}\cdot\partial_t\vec{X})\dot{X}^\mu + (c^2 - (\partial_t\vec{X})^2)\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}},$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{ds}{d\sigma}\frac{\dot{X}^\mu - (\partial_s\vec{X}\cdot\partial_t\vec{X})\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}}.$$

• Free, Neuman BCs, P^σ_μ for the $\mu = 0$ component implies that endpoints move transversely, $\partial_s\vec{X}\cdot\partial_t\vec{X} = 0$, so $\vec{v}_\perp = \vec{v}$. The condition $\vec{P}^\sigma = 0$ at the endpoints implies that the speed of light, $v = c$, for the free (Neuman) BCs.

• Choose σ parameterization such that

$$\partial_\sigma\vec{X}\cdot\partial_\tau\vec{X} = 0 \qquad \text{and} \qquad d\sigma = \frac{ds}{\sqrt{1 - v_\perp^2/c^2}} = \frac{dE}{T_0}.$$

(Using $H = \int T_0 ds/\sqrt{1 - v_\perp^2/c^2}$ and $\partial_t(ds/\sqrt{1 - v_\perp^2/c^2}) = 0$.) The last equation above is equivalent to $(\partial_\sigma\vec{X})^2 + c^{-2}(\partial_t\vec{X})^2 = 1$. With this worldsheet gauge choice,

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\partial_t X^\mu = \frac{T^0}{c^2}(c, \vec{v}_\perp), \qquad \mathcal{P}^{\sigma,\mu} = -T_0\partial_\sigma X^\mu = (0, -T_0\partial_\sigma\vec{X}).$$

The equation of motion is then simply $(\partial_t^2 - c^2\partial_\sigma^2)\vec{X} = 0$.