4/25/12 Lecture outline

⋆ Reading: Zwiebach chapters 4, 5, 6.

• Last time:

$$S = \int_\Sigma d^{d_W}\xi \mathcal{L}(\phi, \partial_\alpha \phi),$$

the variation is

$$\delta S = \int_\Sigma d^{d_W}\xi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi}\right)\delta\phi + \int_{\partial\Sigma} \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi}\delta\phi (d^{d_W-1}\xi)^\alpha,$$

where the last term is the boundary contribution, obtained by integrating a total derivative using Gauss' law. The Euler/Lagrange equations are thus

$$\left(\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\right) = 0,$$

where we included an extra index a to be more general.

We also have to ensure that the boundary term vanishes, which is done by requiring either $\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a} n^\alpha|_{\partial\Sigma} = 0$, where n^α is perpendicular to the boundary, or by requiring that ϕ^a is constant along the boundary, or by a combination of these.

While we're at it, let's recall/quote Noether's theorem, relating continuous symmetries of the action to conservation laws. If $\mathcal{L}$ is invariant under some continuous transformation $\phi^a \to \phi^a + \delta\phi^a$, then there is a conserved quantity j^α:

$$\partial_\alpha j^\alpha = 0 \qquad \text{with} \qquad j^\alpha \sim \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\delta\phi^a,$$

where the conservation law follows from $\delta\mathcal{L} = 0$ and the Euler-Lagrange equations. We'll see that spacetime conservation laws, like spacetime momentum and angular momentum conservation, will arise from such conserved currents on the string worldsheet.

There is also conservation of world-volume energy/ momentum, coming from world-volume translation symmetry, $\xi \to \xi + \delta\xi$: $\partial_\alpha T^{\alpha\beta} = 0$, where

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\partial_\beta \phi^a - g_{\alpha\beta}\mathcal{L}$$

is the world-volume energy momentum tensor.

• Recall $S = -mc\int ds + \frac{q}{c}\int A_\mu dx^\mu$ for a relativistic point particle, where we can write $ds = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}d\tau$, with $\dot{} \equiv \frac{d}{d\tau}$, and τ is the arbitrary worldline parameter, with reparameterization symmetry $\tau \to \tau'$. For a string world-sheet, we need two parameters, ξ^a, $a = 1, 2$. The string trajectory is $x : \Sigma \to M$, where Σ is the 2d world-sheet, with local coordinates ξ^a, and M is the target space, with local coordinates x^μ. The worldsheet area element is $A = \int d^2\xi\sqrt{|h|}$, where h_{ab} is the worldsheet metric, and $|h|$ is its determinant. Suppose that the target space has metric $g_{\mu\nu}$, with space-time length e.g. $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. By writing $dx^\mu = \partial_a x^\mu d\xi^a$, we get

$$ds^2 = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b}d\xi^a d\xi^b, \qquad \text{so} \qquad h_{ab} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b},$$

where this h_{ab} is called the induced metric. So the worldsheet area functional is

$$A = \int d^2\xi\sqrt{\det(g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b})}.$$

• For strings in Minkowski spacetime, we write it instead as $X^\mu(\tau,\sigma)$. There is also a needed minus sign, as the area element is $\sqrt{|g|}$, actually involves the absolute value of the determinant, and the determinant is negative (just like $\det\eta = -1$). So

$$A = \int d\tau d\sigma\sqrt{(\frac{\partial X}{\partial\tau}\cdot\frac{\partial X}{\partial\sigma})^2 - (\frac{\partial X}{\partial\tau})^2(\frac{\partial X}{\partial\sigma})^2},$$

where the spacetime indices are contracted with the metric $g_{\mu\nu}$. To get an action with $[S] = ML^2/T$, we have

$$S_{Nambu-Goto} = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau\int d\sigma\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

where we define $\dot{X}^\mu \equiv \frac{dx^\mu}{d\tau}$ and $X\mu' \equiv \frac{\partial X^\mu}{\partial\sigma}$ annd T_0 is the string tension, with $[T_0] = [F] = [ML/T^2]$.

The action is reparameterization invariant: can take $(\tau,\sigma) \to (\tau'(\tau,\sigma),\sigma'(\tau,\sigma))$ and get $S \to S$. Enormous symmetry/redundancy in choice of (τ,σ); can "fix the gauge" to some convenient choice, and the physics is completely independent of the choice.

• We can write S_{NG} in terms of the Lagrangian density

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial\mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

For the open string, $\delta S = 0$ also requires $\int d\tau [\delta X^\mu P^\sigma_\mu]_0^{\sigma_0} = 0$, which requires for each μ index either of the Dirichlet or Neumann BCs, at each end:

$$\text{Dirichlet} \qquad \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \qquad \rightarrow \qquad \delta X^\mu(\tau, \sigma_*) = 0,$$

$$\text{Neumann} \qquad \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0.$$