4/23/12 Lecture outline

$\star$ Reading: Zwiebach chapters 4, 5, 6.

• Nonrelativistic strings. $[T_0]=[F]=[E]/L=[\mu_0][v^2]$. Indeed, considering $F=ma$ for an element dx of the string yields the string wave equation $\frac{\partial^2 y}{\partial x^2}-\frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2}=0$, with $v_0=\sqrt{T_0/\mu_0}$. Endpoints at $x=0$ and $x=a$. Can choose Dirichlet or Neumann BCs at these points. With Dirichlet at each end, $y_n(x)=A_n\sin(n\pi x/a)$ and the general solution is $y(x,t)=\sum_n y_n(x)\cos\omega_n t$, where $\omega_n=v_0 n\pi/a$ (and the A_n are determined from the initial conditions, by Fourier transform).

The nonrelativistic string action is $S=\int dtL$ where L is the kinetic energy minus potential energy, which gives

$$S=\int dt\int dx\left(\tfrac{1}{2}\mu_0(\frac{\partial y}{\partial t})^2-\tfrac{1}{2}T_0(\frac{\partial y}{\partial x})^2\right),$$

which is a particular case of the more general action $S=\int dtdx\mathcal{L}(\frac{\partial y}{\partial t},\frac{\partial y}{\partial x})$. We can then define the momentum density and corresponding spatial quantity

$$\mathcal{P}^t=\frac{\partial\mathcal{L}}{\partial\dot y},\qquad \mathcal{P}^x=\frac{\partial\mathcal{L}}{\partial y'}.$$

The variation of the action is

$$\delta S=\int dtdx[\mathcal{P}^t\delta\dot y+\mathcal{P}^x\delta y']=-\int dtdx[\frac{\partial\mathcal{P}^t}{\partial t}+\frac{\partial\mathcal{P}^x}{\partial x}]\delta y+\text{bndy terms}$$

and the action is made stationary, $\delta S=0$, if the boundary terms vanish and if

$$\frac{\partial\mathcal{P}^t}{\partial t}+\frac{\partial\mathcal{P}^x}{\partial x}=0,$$

which when applied to the above particular choice of action gives the usual wave equation. The boundary terms must also be set to zero, and they involve $\mathcal{P}^t\delta y$ at the time endpoints and $\mathcal{P}^x\delta y$ at the space endpoints. Neumann BCs is to set $\mathcal{P}^x=0$ at the spatial endpoints (for all t), and Dirichlet BCs is to set $\delta y=0$ (and thus $\mathcal{P}^t=0$) at the spatial endpoints.

• Particle $q(\tau)$ vs field $\phi(\xi^\alpha)$ for $\alpha=0,\dots d_W-1$: particle is the case of a single ξ, $d_W=1$, vs more than one for a field (e.g. $\vec E(t,\vec x)$). Fields have

$$S=\int_\Sigma d^{d_W}\xi\mathcal{L}(\phi,\partial_\alpha\phi),$$

the variation is

$$\delta S=\int_\Sigma d^{d_W}\xi\left(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\alpha\phi}\right)\delta\phi+\int_{\partial\Sigma}\frac{\partial\mathcal{L}}{\partial\partial_\alpha\phi}\delta\phi(d^{d_W-1}\xi)^\alpha,$$

where the last term is the boundary contribution, obtained by integrating a total derivative using Gauss' law. The Euler/Lagrange equations are thus

$$\left(\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\right) = 0,$$

where we included an extra index a to be more general.

We also have to ensure that the boundary term vanishes, which is done by requiring either $\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a} n^\alpha|_{\partial \Sigma} = 0$, where n^α is perpendicular to the boundary, or by requiring that ϕ^a is constant along the boundary, or by a combination of these.