4/18/12 Lecture outline

⋆ Reading: Zwiebach chapters 2, 3 and 5.

• Last time: in quantum mechanics, derivatives only appear in the "covariant derivative" combination

$$D_\mu = \partial_\mu - i\frac{q}{\hbar c}A_\mu. \tag{1}$$

This is crucial for gauge invariance of physics under $A_\mu \to A_\mu + \partial_\mu f$, as the wavefunction changes under gauge transformations:

$$\psi(t,\vec{x}) \to e^{iqf(t,\vec{x})/\hbar c}\psi(t,\vec{x}). \tag{2}$$

The above covariant derivatives have the property that $D_\mu\psi \to e^{iqf/\hbar c}D_\mu\psi$ under a gauge transformation, with the shift $A_\mu \to A_\mu + \partial_\mu f$ canceling the bad term $\sim \partial_\mu f$. Because derivatives are all covariant, the local parameter $f(x)$ always only enters as an overall phase, which remains physically unobservable upon computing probability $||\cdot||^2$.

Gauge invariance says that physics observables can't notice gauge transformations by arbitrary $f(x)$. This phase transformation is called $U(1)$ gauge invariance, i.e. we can take $\psi \to U(x)\psi$, where $U(x) = e^{iqf/\hbar c}$ is an arbitrary local $U(1)$ symmetry transformation. This is why electromagnetism is called a $U(1)$ gauge theory in modern high energy physics, where gauge symmetries are fundamental, and in direct correspondence with the fundamental forces. Each of the 4-known forces is associated with a gauge invariance. (Gravity's is general coordinate invariance.)

The $U(1)_{EM}$ symmetry is the symmetry of rotating a circle. In Kaluza-Klein theory, this circle is that of the compact 5-th dimension! Since charge is quantized, $q = ne$, where $-e$ is the charge of an electron, the gauge symmetry above doesn't even change the wavefunction if $f \to f + 2\pi\hbar c/e$

Another idea for charge quantization: monopoles and Dirac quantization. In vacuo, Maxwell's equations are symmetric under $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ (in relativistic notation, $F^{\mu\nu} \to \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$). Dirac string and $e^{iS/\hbar} \to e^{iS/\hbar} + e^{ie\oint \vec{A}\cdot d\vec{x}/\hbar c}$ so $\oint \vec{A}\cdot d\vec{x} = \int \vec{B}\cdot d\vec{a} = 4\pi g$ where $eg = \frac{1}{2}\hbar c n$, with n an integer. We haven't seen a monopole yet, but inflation could have removed them. Also in GUTs, $U(1)_{EM}$ is part of a larger symmetry, which leads to monopoles and charge quantization.

• A brief (!) introduction to general relativity. We replace the metric $\eta_{\mu\nu}$ with a dynamical quantity $g_{\mu\nu}$. There is a symmetry principle which is akin to the gauge invariance of electricity and magnetism and to the above repraparameterization invariance.

It is general coordinate invariance: $x^\mu \to x^{\mu'}(x^\mu)$. Physics is invariant under such local coordinate changes. The metric transforms as $g_{\mu\nu} = g_{\mu'\nu'}\frac{dx^{\mu'}}{dx^\mu}\frac{dx^{\nu'}}{dx^\nu}$. The action of a point particle is $S = -mc\int ds + \frac{q}{c}\int A_\mu dx^\mu$, just like before, except that we contract and raise and lower indices with $g_{\mu\nu}$ rather than $\eta^{\mu\nu}$. Get from the Euler Lagrange equations now

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{q}{mc}F^\mu_\nu\frac{dx^\nu}{d\tau},$$

where

$$\Gamma^\mu_{\rho\sigma} = \tfrac{1}{2}g^{\mu\lambda}(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma})$$

is the connection; it is analogous to A^μ in electromagnetism. The connection enters into covariant derivatives like $\nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\rho\sigma}V^\sigma$ in order to have things transform properly under general coordinate transformations (analogous to the gauge invariant covariant derivatives $D_\mu = \partial_\mu - i\frac{q}{\hbar c}A_\mu$. in E&M). The above equations of motion is called the geodesic equation; it is reparameterization invariant ($\tau \to \tau'$) and transforms properly under general coordinate transformations $x^\mu \to x^{\mu'}$.

The Riemann tensor is

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu).$$

It is analogous to $F_{\mu\nu}$ in E&M. The Ricci tensor is $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ and the Ricci scalar is $R = R^\mu_\mu$. The metric is dynamically determined by minimizing the action w.r.t. $\delta g_{\mu\nu}$, where there is a term

$$S = \frac{1}{16\pi G_D}\int d^Dx\sqrt{|g|}R + \ldots.$$

For fun, we wrote it in general spacetime dimension D. Let's note the units (setting c=1): $[R] = L^{-2}$ and $[S] = ML$, so $[G_D] = L^{D-3}M^{-1}$. Since $[\hbar] = ML$, we have $G_D = \ell_P^{D-2}$ in D spacetime dimensions. (Note that $\int d^Dx\sqrt{|g|}$ gives the spacetime volume (which is clearly general coordinate invariant). This comment will be useful very soon, when we write down the relativistic string action!)

Note also that the relation $G_D = GV_C$ is evident from the above action.

In the weak curvature limit, we can reduce to the gravitational potentials, with $\nabla^2 V_g^{(D)} = 4\pi G_D\rho_m$. This comes from $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ and $h_{0,0} \approx -2V_g$.

• Electromagnetism in other dimensions: $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$ and $\partial_\mu F^{\mu\nu} = \frac{1}{c}j^\nu$. So e.g. a point charge q makes an electric field with $\nabla\cdot\vec{E} = q\delta^d(\vec{x})$ in a world with $D = d+1$ spacetime dimension (the +1 is the time dimension, and there are d spatial directions), so $\int_{S^{d-1}} \vec{E} \cdot d\vec{a} = q$. Thus $\vec{E} = E(r)\hat{r}$ with $E(r) = q/r^{d-1} vol(S^{d-1})$, where $vol(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit sphere surrounding the charge. Finally, we get that a point charge makes electric field given by $E(r) = \Gamma(d/2) q / 2\pi^{d/2} r^{d-1}$. For $d = 3$, get $E(r) = q/4\pi r^2$, which is the usual answer in these units.

• Gravity has general coordinate invariance, $x^\mu \to x^{\mu'}(x^\mu)$. At the linearized level, take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $x^{\mu'} = x^\mu + \epsilon^\mu(x)$, with $\delta h_{\mu\nu} \approx \partial_{(\mu}\epsilon_{\nu)}$.

Recall $m_P = \sqrt{\hbar c/G} = 2.176 \times 10^{-5} g$.

In 4d, we have gravitational potential given by $V_g^{(4)} = -GM/r$, which solves $\nabla^2 V_g^{(D)} = 4\pi G^{(D)} \rho_m$. This is taken to be the gravitational potential equation in any spacetime dimension, with gravitational force taken to be $F = -m\nabla V_g$. In $\hbar = c = 1$ units, get $G = \ell_P^{D-2}$ in D spacetime dimensions. Get $G^D = G V_C$, where V_C is the compactification volume.

$\ell_C = \ell_P^{(D)} (\ell_P^{(D)}/\ell_P)^{2/(D-4)}$, can imagine e.g. $\ell_P^{(D)} \sim 10^{-18} cm$ instead of $\ell_P \sim 10^{-33} cm$ (i.e. lower gravitational physics effects to $M_P^{(D)} \sim 20 TeV$ from $M_P \sim 10^{16} TeV$) which for $D = 6$ gives $\ell_C \sim 10^{-3} cm$ – large extra dimensions.