4/16/12 Lecture outline

⋆ Reading: Zwiebach chapters 2 and 3.

• As mentioned last time, the action for a relativistic point particle of mass m is $S = -mc\int ds = -mc^2\int dt\sqrt{1-v^2/c^2}$. This gives $\vec{p} = \partial_{\vec{v}} = \gamma m\vec{v}$ and $H = \vec{p}\cdot\vec{v} - L = \gamma mc^2$, both of which are constants of the motion (thanks to the time and spatial translation invariance).

• Reparametrization invariance: write $x_\mu(\tau)$, and can change worldline parameter τ to an arbitrary new parameterization $\tau'(\tau)$, and the action is invariant. To see this use $S = -mc\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ and change $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}$ and note that $S\to S$. The Euler Lagrange equations of motion are $\frac{dp_\mu}{d\tau} = 0$.

When the particle is charged and in the presence of electric and magnetic fields, there is the new term in the action

$$S = \int(-mcds + \frac{q}{c}A_\mu dx^\mu), \tag{1}$$

which is manifestly relativistically invariant (and also repparameterization) invariant. Note also that, under a gauge transformation, we have $S\to S+\frac{qf}{c}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L = -mc\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P} = \partial L/\partial\vec{v} = m\vec{v}/\sqrt{1-\vec{v}^2/c^2} + \frac{q}{c}\vec{A}$. The Hamiltonian is $H = \vec{v}\cdot\vec{P} - L = \sqrt{m^2c^4 + c^2(\vec{P}-\frac{q}{c}\vec{A})^2} + q\phi$.

The equations of motion can be written as $\frac{d^2x^\mu}{d\tau^2} = \frac{q}{mc}F_{\mu\nu}\frac{dx^\nu}{d\tau}$.

In the non-relativistic limit we have $H = \frac{1}{2m}(\vec{P}-\frac{q}{c}\vec{A})^2 + q\phi$, where $\vec{P}-\frac{q}{c}\vec{A} = m\vec{v}$.

• The electric and magnetic fields themselves have a lagrangian, with action

$$S = \int d^4x\mathcal{L}, \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}A_\mu j^\mu.$$

The two Maxwell's equations expression absence of magnetic monopoles are, again, solved by setting $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$. The other two Maxwell's equations then come from the Euler -Lagrange equations of the above action upon varying $A_\mu \to A_\mu + \delta A_\mu$: the action is stationary when

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} - \frac{\partial\mathcal{L}}{\partial A_\mu} = 0.$$

• In quantum mechanics, we have $[x^k, P^\ell] = i\hbar\delta^{k,\ell}$, so we replace $\vec{P} \to -i\hbar\nabla$ in position space. The S.E. is $H\psi = i\hbar\partial_t\psi$. Note that the derivatives only appear in the "covariant derivative" combination

$$D_\mu = \partial_\mu - i\frac{q}{\hbar c}A_\mu. \tag{2}$$

This is crucial for gauge invariance of physics.

The reason is that gauge invariance is more interesting in quantum theory, as the wavefunction changes under gauge transformations:

$$\psi(t,\vec{x}) \to e^{iqf(t,\vec{x})/\hbar c}\psi(t,\vec{x}), \tag{3}$$

which leaves the probability density and current unchanged. (One way to see this is via $\psi \sim e^{iS/\hbar}$ and noting from the above expression for S that that $S \to S + qf/c$.)

The above covariant derivatives have the property that $D_\mu\psi \to e^{iqf/\hbar c}D_\mu\psi$ under a gauge transformation, with the shift $A_\mu \to A_\mu + \partial_\mu f$ canceling the bad term $\sim \partial_\mu f$. Because derivatives are all covariant, the local parameter $f(x)$ always only enters as an overall phase, which remains physically unobservable upon computing probability $||\cdot||^2$.

Gauge invariance says that physics observables can't notice gauge transformations by arbitrary $f(x)$. This phase transformation is called $U(1)$ gauge invariance, i.e. we can take $\psi \to U(x)\psi$, where $U(x) = e^{iqf/\hbar c}$ is an arbitrary local $U(1)$ symmetry transformation. This is why electromagnetism is called a $U(1)$ gauge theory in modern high energy physics, where gauge symmetries are fundamental, and in direct correspondence with the fundamental forces. Each of the 4-known forces is associated with a gauge invariance. (Gravity's is general coordinate invariance.)

According to Noether's theorem, there is a one-to-one correspondence

$$\text{(continuous) global symmetry} \leftrightarrow \text{conserved quantity}.$$

The original example the relation between translation symmetry in time and/or space, $x^\mu \to x^\mu + a^\mu$, and conservation of energy and/or momentum, p^μ.

There is a deep correspondence

$$\text{local gauge symmetry} \to \text{forces}.$$

and E&M is the force associated with the local symmetry above. There is still a conserved charge, in E&M it is current conservation $\partial^\mu j_\mu = 0$. As we'll now discuss, in general relativity (GR) the above spacetime translation symmetry is a subgroup of a more general symmetry, general coordinate invariance, which is the fundamental symmetry principle associated with gravity.