6/6/12 Lecture outline

$\star$ Reading: Zwiebach chapter 17 and 22.

• Finish up from last time. For closed superstrings we can take the $NS+$ sector for both left and right movers, and the $R-$ sector for both left and right movers; this is the IIB superstring. Or we could take the $NS+$ sector for both left and right movers, and the $R-$ sector for left movers and the $R+$ sector for right movers; this is the IIA superstring.

The massless (NS+, NS+) states for both of these string theories consist of

$$\widetilde{b}^I_{-\frac{1}{2}}|NS\rangle_L \otimes b^J_{-\frac{1}{2}}|NS\rangle_R \otimes |p\rangle.$$

As in the bosonic case, these correspond to $g_{\mu\nu}$,, $B_{\mu\nu}$, ϕ.

• Consider the closed, bosonic string on a circle, $X_{25} \sim X_{25} + 2\pi R$. If we were dealing with particles rather than strings, we know what would happen: the momentum in the circle direction is quantized (by $\psi \sim e^{ip\cdot x}$ being set equal to itself when going around the circle) as

$$p_{25} = \frac{n}{R}, \qquad n = 0, \pm 1, \pm 2 \dots.$$

For a big circle, these are closely spaced together, and for a small circle they are widely separated. That's why it's hard to experimentally rule out the absence of tiny, rolled up, extra dimensions: it could just take more energy than we can make presently to excite one of the $n \neq 0$ "Kaluza-Klein modes."

Now we're going to describe something bizarre about strings: there is a symmetry, called T-dualtiy, which makes the physics invariant under $R \leftrightarrow \alpha'/R$. This is strange: a very big circle is physically indistinguishable from a very small circle! The reason is that, in addition to momentum, there are string winding modes, and T-duality exchanges them. For a big circle, the momentum modes are light and the winding modes are heavy, and for a tiny circle they're reversed, but same physics. Smallest possible effective distance, $R = \sqrt{\alpha'}$.

The winding number is given by $X(\tau, \sigma + 2\pi) - X(\tau, \sigma) = m(2\pi R)$. We then have $X = X_L + X_R$ with

$$X_L(\tau + \sigma) = const. + \tfrac{1}{2}\alpha'(p + w)(\tau + \sigma) + \text{oscillators},$$

$$X_R(\tau - \sigma) = const + \tfrac{1}{2}\alpha'(p - w)(\tau - \sigma) + \text{oscillators}.$$

Here

$$p = \frac{n}{R}, \qquad w = \frac{mR}{\alpha'}.$$

The T-duality symmetry comes from the symmetry $(p_L, p_R) \to (p_L, -p_R)$, where

$$p_L = \frac{n}{R} + \frac{mR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{mR}{\alpha'}.$$

Also, to have $X(\tau, \sigma + 2\pi) \sim X(\tau, \sigma) + 2\pi R m$, we need $N^\perp - \widetilde{N}^\perp = nm$.

• Let's now consider some aspects of string thermodynamics. The number of string states grows very rapidly with excitation number, and it turns out that this puts an upper limit on the temperature, beyond which the partition function would not converge.

Before getting into thermodynamics, let's count string states. Recall that we counted states of the open bosonic string via

$$f_{os}(x) = \mathrm{Tr}_{states} x^{\alpha' M^2} = \left(\frac{1}{x^{1/24} \prod_{n=1}^\infty (1 - x^n)} \right)^{(D-2)} \equiv \sum_{N=0}^\infty p_{D-2}(N) x^{N-(D-2)/24}.$$

We saw that $D = 26$, but let's keep it as a parameter for the moment. Here $p_{D-2}(N)$ is the number of distinct partitions of N into arbitrary numbers of non-negative integers, each of which can have $D-2$ labels. This corresponds to how many choices of $\lambda_{I,n}$ there are such that $\prod_{I=2}^{D-1} \prod_{n=1}^\infty (a_n^{I\dagger})^{\lambda_{I,n}}$ has $N^\perp = \sum_n \sum_I n \lambda_{I,n} = N$. Let's consider $p_1(N) = p(N)$ as an illustration: $p(5) = 7$, $p(10) = 42$, find $p(N)$ grows rapidly with N. The large N behavior of $p(N)$ was studied long ago by number theorists Hardy and Ramanujan:

$$p(N \gg 1) \approx \frac{1}{4N\sqrt{3}} \exp(2\pi \sqrt{\frac{N}{6}}).$$

Can also show:

$$p_b(N \gg 1) \approx \frac{1}{\sqrt{2}} \left(\frac{b}{24} \right)^{(b+1)/2} N^{-(b+3)/4} \exp(2\pi \sqrt{\frac{Nb}{6}}).$$

Note the appearance of 24 in this number theory formula, which will be related to $D - 2 = 24$ in string theory. In fact, these formula were derived by relating the above generating functions to the Dedekind eta function:

$$\eta(\tau) \equiv e^{i\pi\tau/12} \prod_{n=1}^\infty (1 - e^{2\pi i n \tau}).$$

Note that this function, defined long ago by mathematicians, nicely allows us to write

$$f_{os}(x = e^{2\pi i \tau}) = \eta(\tau)^{-(D-2)}.$$

An important property of the eta function (both for math, and for string theory!) is

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau),$$

and this allows us to relate the $x \to 1$ limit, which is relevant for extracting $p(N \gg 1)$, to another limit: $x \to 1$ is $\tau \to 0i$, which can be related to $-1/\tau \to i\infty$.

Setting $\tau = i\tau_2$, the above generating functions start to resemble partition functions, with $H = \alpha' M^2$. (Actually, they are partition functions, but on the worldsheet for the moment.)

$$f_{os} = \mathrm{Tr}_{states} e^{-2\pi\tau_2\alpha' M^2} = \sum_{N=0}^{\infty} p_{D-2}(N) e^{-2\pi\tau_2(N-1)}.$$

For large N, we have $M^2 \approx N/\alpha'$ and, taking $E = M$, we have $\sqrt{N} = \sqrt{\alpha'}E$. The entropy of string states with energy E is then

$$S(E) = k\ln\Omega(E) = k\ln p_{24}(N = \sqrt{\alpha' E}) \approx k 4\pi\sqrt{\alpha'}E.$$

Then

$$\frac{1}{kT} = \frac{1}{k}\frac{\partial S}{\partial E} = 4\pi\sqrt{\alpha'} \equiv \frac{1}{kT_H},$$

where T_H is the Hagedorn temperature.

To fully compute the spacetime partition function, we must write $E = \sqrt{\vec{p}^2 + M^2}$ and do the integral over momentum, $V\int d^{D-1}p/(2\pi\hbar)^{D-1}$. Find that $Z_{string}(T)$ has a pole as $T \to T_H$: $Z_{string} \sim C/(T - T_H)$, with C a constant.

- Black holes. Discuss $R = 2GM$.
- Black hole thermodynamics:

$$S_{B.H.}/k = \frac{A}{4},$$

works in any dimension, for any black hole, where A is measured in Planck units. E.g. in 4d,

$$S/k = \frac{4\pi R^2}{4\ell_P^2} = \frac{4\pi R^2}{4\hbar G_N/c^3} = \frac{4\pi G}{\hbar c}M^2.$$

Yields $kT_{BH} = \hbar c^3/8\pi GM$. Discuss Hawking radiation, information puzzle.

- Holography.
- Modern string theory. Dualities. Issues.