6/4/12 Lecture outline

⋆ Reading: Zwiebach chapter 14.

• Last time: The closed string states are given by acting with left and right moving creation operators on $|p^+, p^I\rangle$, with the constraint that $N^\perp = \widetilde{N}^\perp$ (because of translation symmetry in shifting σ). In summary, the spectrum of states is given by

$$
\begin{gathered}
|\lambda, \widetilde{\lambda}\rangle = [\prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(a_n^{I\dagger})^{\lambda_{n,I}}][\prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(\widetilde{a}_n^{I\dagger})^{\widetilde{\lambda}_{n,I}}]|p^\mu\rangle \\
M^2 = -p^2 = 2(N_\perp + \widetilde{N}_\perp - 2)/\alpha', \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\lambda_{n,I}, \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\widetilde{\lambda}_{n,I},
\end{gathered} \tag{1}
$$

where there is a requirement that $N^\perp = \widetilde{N}^\perp$ to have σ translation invariance.

The state with $N^\perp = 0$ is the bosonic closed string tachyon. Those with $N^\perp = 1$ are given by a $(D-2)^2$ matrix of indices in the transverse directions, and these are massless. The symmetric traceless part is the graviton, the antisymmetric tensor is a gauge field $B_{\mu\nu}$ which is an analog of A_μ, and the trace part is ϕ, called the "dilaton."

Count the states by defining $f(x) = \mathrm{Tr}_{states} x^{\alpha' M^2}$. Find

$$
f_{os}(x) = x^{-1}\prod_{n=1}^{\infty}\frac{1}{(1-x^n)^{24}}
$$

where we set $D-2=24$. Similarly, for the closed string case, we have

$$
f_{closed}(x,\bar{x}) = f_{os}(x)f_{os}(\bar{x}),
$$

where we need to project out those states with different powers of x and $\bar{x}$.

• Superstrings! The bosonic string has fields $X^I(\tau,\sigma)$, which are $D-2$ worldsheet scalars. Now we introduce $D-2$ worldsheet fermions

$$
\Psi_R(\tau-\sigma)^I, \qquad \Psi_L^I(\tau+\sigma).
$$

Here R and L are for right and left moving, and $I = 2\ldots D-2$ (spacetime vector indices). The light cone gauge action is

$$
S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(\dot{X}^I\dot{X}^I - X^{I'}X^{I'} + \Psi_R^I(\partial_\tau+\partial_\sigma)\Psi_R^I + \Psi_L^I(\partial_\tau-\partial_\sigma)\Psi_L^I\right).
$$

Note that the bosons X^I have the usual quadratic in derivatives terms (like $L = \frac{1}{2}m\dot{x}^2$) whereas the fermions Ψ have linear in derivatives terms. The fermion and its action can roughly be thought of as the square-root of a boson and its action. The terms in the action above is the 2d worldsheet version of the Dirac equation action. The classical equations of motion from the Euler Lagrange equations are just $(\partial_\tau+\partial_\sigma)\Psi_R=0$ and $(\partial_\tau-\partial_\sigma)\Psi_L=0$, which are solved by $\Psi_R=\Psi_R(\tau-\sigma)$ and $\Psi_L=\Psi_L(\tau+\sigma)$.

There are two choices of boundary conditions for left movers, and similarly two choices for right movers:

$$\Psi^I(\tau,\sigma+2\pi)=\pm\Psi^I(\tau,\sigma), \qquad +:\text{Ramond}, \quad -:\text{Nevu-Schwarz}.$$

In the NS sector we have

$$\Psi^I_{NS}\sim\sum_{n=-\infty}^{\infty}b^I_{n+\frac{1}{2}}e^{-i(n+\frac{1}{2})(\tau-\sigma)}.$$

In the R sector we have

$$\Psi^I_R\sim\sum_{n=-\infty}^{\infty}d^P_n e^{-in(\tau-\sigma)}.$$

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$$\Psi^I_R\sim\sum_{n=-\infty}^{\infty}d^P_n e^{-in(\tau-\sigma)}.$$

The modes satisfy

$$\{b^I_r,b^J_s\}=\delta_{r+s,0}\delta^{IJ}, \qquad \{d^I_n,d^J_m\}=\delta_{n+m,0}\delta^{IJ},$$

where $\{A,B\}\equiv AB+BA$ is the anti-commutator, reflecting the fermionic nature of the modes.

The NS sector states are

$$|\lambda,\rho\rangle_{NS}=\prod_{I=2}^{D-2}(a^{I\dagger}_n)^{\lambda_{n,I}}\prod_{J=2}^{D-1}\prod_{r=\frac{1}{2},\frac{3}{2}\cdots}(b^J_{-r})^{\rho_{r,J}}|NS\rangle\otimes|p\rangle,$$

where the $\rho_{r,J}$ are either zero or one (Fermi statistics).

The R sector states are

$$|\lambda,\rho\rangle_R = \prod_{I=2}^{D-2}\prod_n (a_n^{I\dagger})^{\lambda_{n,I}} \prod_{J=2}^{D-1}\prod_{m=1}^{\infty}(d_{-m}^J)^{\rho_{m,J}}|R_A\rangle\otimes|p\rangle,$$

The upshot in this case is that $D=10$ spacetime dimensions is needed. The mass-squared operator in the NS sector is

$$\alpha' M^2 = N^\perp + \tfrac{1}{2}(D-2)(-\frac{1}{12}-\frac{1}{24}),$$

where the $-1/12$ was seen in the bosonic case, and the $-1/24$ is the analog coming from reordering the b_r. As in the bosonic case, the commutator $[M^{-I}, M^{-J}]=0$ determines the spacetime dimension, here to be $D=10$. Similarly, in the R-sector, we have

$$\alpha' M^2 = N^\perp, \qquad N^\perp = \sum_{p=1}^{\infty} p a_p^{\dagger I} a_p^I + \sum_{m=1}^{\infty} m d_{-m}^I d_m^I.$$

The NS spectrum generating function is

$$f_{NS}(x) = \frac{1}{\sqrt{x}}\prod_{n=1}^{\infty}\left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\right)^8 .$$

The R sector spectrum generating function is

$$f_{R\pm}(x) = 8\prod_{n=1}^{\infty}\left(\frac{1+x^n}{1-x^n}\right)^8$$

where 8 accounts for the ground state degeneracy associated with d_0^I, in either the R_+ or the R_- sector. We should also GSO project the NS sector, i.e. throw away states with $(-1)^F=-1$ to get the $NS+$ states, with generating function

$$f_{NS+}(x) = \frac{1}{2\sqrt{x}}\left[\prod_{n=1}^{\infty}\left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n}\right)^8 - \left(\frac{1-x^{n-\frac{1}{2}}}{1-x^n}\right)^8\right].$$

This projects out the tachyon – nice! Moreover, the states in $f_{R\pm}$ are spacetime fermions, whereas those in $f_{NS,+}$ are spacetime bosons, and their spectrum is degenerate, thanks to the identity $f_{R\pm}(x)=f_{NS+}(x)$ (which was proven as a mathematical identity around 150 years before the superstring was even first thought of!).

• For closed superstrings we can take the $NS+$ sector for both left and right movers, and the $R-$ sector for both left and right movers; this is the IIB superstring. Or we could take the $NS+$ sector for both left and right movers, and the $R-$ sector for left movers and the $R+$ sector for right movers; this is the IIA superstring.

The massless (NS+, NS+) states for both of these string theories consist of

$$\widetilde{b}^I_{-\frac{1}{2}}|NS\rangle_L \otimes b^J_{-\frac{1}{2}}|NS\rangle_R \otimes |p\rangle.$$

As in the bosonic case, these correspond to $g_{\mu\nu}$,, $B_{\mu\nu}$, ϕ.