5/30/12 Lecture outline

$\star$ Reading: Zwiebach chapter 13

• Recall from last time that we considered the open string in light cone gauge. So we fixed $X^+$ to be simply related to $\tau$, found that the $X^I$ are given by simple harmonic oscillators, and $X^-$ is a complicated expression, fully determined in terms of the transverse direction quantities:

$X^+(\tau,\sigma) = 2\alpha' p^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau$. For $X^-$ recall expansion, with $\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp$, where $L_n^\perp \equiv \frac{1}{2}\sum_p \alpha_{n-p}^I\alpha_p^I$ is the transverse Virasoro operator. Recall $[\alpha_m^I, \alpha_n^J] = m\delta^{IJ}\delta_{m+n,0}$ and accounting for the ordering of $L_0^\perp$ was needed.

The upshot was that the relativistic open string spectrum is given by

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(a_n^{I\dagger})^{\lambda_{n,I}}|p^\mu\rangle \quad \text{with} \quad M^2 = -p^2 = (N_\perp - 1)/\alpha', \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\lambda_{n,I}. \quad (1)$$

Where consistency requires $D = 26$.

• Now consider closed string case. Recall gauge conditions $n\cdot X = \alpha'(n\cdot p)\tau$, $n\cdot p = 2\pi n\cdot \mathcal{P}^\tau$, which yielded the constraints $(\dot X \pm X')^2 = 0$ and then the EOM were simply $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$. For the closed string, this means that $X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)$. The general solutions can then be written as

$$X_R^\mu(v) = \tfrac{1}{2}x_0^{L\mu} + \sqrt{\tfrac{1}{2}\alpha'}\alpha_0^\mu v + i\sqrt{\tfrac{1}{2}\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-inv},$$

and a similar expression for $X_L^\mu$, with modes $\widetilde{\alpha}_n^\mu$. Since $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$, $\widetilde{\alpha}_0^\mu = \alpha_0^\mu$. Computing $\mathcal{P}^{\mu\mu} = \dot X^\mu/2\pi\alpha'$ then yields $\alpha_0^\mu = \sqrt{\frac{1}{2}\alpha'}p^\mu$.

The theory is quantized by taking $[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')] = -\delta(\sigma-\sigma')\eta^{IJ}$, which implies that

$$[\alpha_m^I, \alpha_n^J] = m\delta_{m+n,0}\eta^{IJ}, \qquad [\widetilde{\alpha}_m^I, \widetilde{\alpha}_n^J] = m\delta_{m+n,0}\eta^{IJ}$$

with no commutator between the left and right movers. It's now very similar to the open string case, but with the two sets of decoupled oscillators for the left and right movers. We define

$$(\dot X^I + X'^I)^2 \equiv 4\alpha'\sum_n \widetilde{L}_n^\perp e^{-in(\tau+\sigma)},$$

and a similar expansion for $(\dot X^I - X'^I)^2$ and $L_n^\perp$, involving $\tau - \sigma$. Then $L_n^\perp = \frac{1}{2}\sum_p \alpha_p^I\alpha_{n-p}^I$, and $L_0^\perp = \frac{\alpha'}{4}p^Ip^I + N^\perp$. The $X^-$ are given in terms of these much as in the open string case, $\sqrt{2\alpha'}\alpha_n^- = 2L_n^\perp/p^+$, with a similar expression for the left movers. The worldsheet Hamiltonian is $H = L_0^\perp + \widetilde{L}_0^\perp - 2$ and $M^2 = -p^2 = 2p^+p^- - p^I p^I = \frac{2}{\alpha'}(N^\perp + \widetilde{N}^\perp - 2)$.

The closed string states are given by acting with left and right moving creation operators on $|p^+, p^I\rangle$, with the constraint that $N^\perp = \widetilde{N}^\perp$ (because of translation symmetry in shifting $\sigma$). In summary, the spectrum of states is given by

$$
\begin{aligned}
&|\lambda, \widetilde{\lambda}\rangle = [\prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(a_n^{I\dagger})^{\lambda_{n,I}}][\prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(\widetilde{a}_n^{I\dagger})^{\widetilde{\lambda}_{n,I}}]|p^\mu\rangle \\
&M^2 = -p^2 = 2(N_\perp + \widetilde{N}_\perp - 2)/\alpha', \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\lambda_{n,I}, \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\widetilde{\lambda}_{n,I},
\end{aligned}
\tag{2}
$$

where there is a requirement that $N^\perp = \widetilde{N}^\perp$ to have $\sigma$ translation invariance.

The state with $N^\perp = 0$ is the bosonic closed string tachyon. Those with $N^\perp = 1$ are given by a $(D-2)^2$ matrix of indices in the transverse directions, and these are massless. The symmetric traceless part is the graviton, the antisymmetric tensor is a gauge field $B_{\mu\nu}$ which is an analog of $A_\mu$, and the trace part is $\phi$, called the "dilaton."

- Let's count the states by defining $f(x) = \mathrm{Tr}_{states} x^{\alpha' M^2}$. Find

$$f_{os}(x) = x^{-1}\prod_{n=1}^{\infty}\frac{1}{(1-x^n)^{24}}$$

where we set $D-2 = 24$. Similarly, for the closed string case, we have

$$f_{closed}(x, \bar{x}) = f_{os}(x) f_{os}(\bar{x}),$$

where we need to project out those states with different powers of $x$ and $\bar{x}$.