5/9/12 Lecture outline

⋆ Reading: Zwiebach chapter 9.

• Recall light cone: $a^\pm = (a^0 \pm a^1)/\sqrt{2}$, so $a\cdot b = -a^- a^+ - a^+ a^- + \sum_I a^I b^I$, where $I = 2, \ldots$ runs over the transverse space directions. Ugly, but can help quantize.

Example from QM: in non-relativistic case, get Schrodinger equation by writing $H = \vec{p}^2/2m + V(x)$ and replacing $H \to i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \to -i\hbar\nabla$. In relativistic case, considering free particle for simplicity, have $H = \sqrt{(c\vec{p})^2 + (mc^2)^2}$ and $\vec{p} \to -i\hbar\nabla$ would require understanding how to take the square-root of an operator. This is what led Dirac to the Dirac equation for relativistic electrons, and the start of quantum field theory. Very interesting and long story, but not the topic of this class. We will avoid going there by the trick of the light cone.

Write $-p\cdot p = m^2$ (setting $c = 1$) as $2p^+p^- = \sum_I p^I p^I + m^2$. In the light cone, we think of x^+ as time. Then $p^- = H_{lc} = (\sum_I p^I p^I + m^2)/2p^+$. No need for square-root. Looks similar to non-relativistic case.

• Recall

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial\mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial\mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}.$$

which we simplified by picking static gauge.

• Generalize static gauge (to eventually get to light cone gauge). Consider e.g. gauge $n_\mu X^\mu = \lambda\tau$ for time-like n_μ (it's orthogonal to the string at constant τ). More generally, can pick

$$n\cdot\mathcal{P}^\sigma = 0, \qquad n\cdot X = \beta\alpha'(n\cdot p)\tau, \qquad n\cdot p = \frac{2\pi}{\beta}n\cdot\mathcal{P}^\tau,$$

where $\beta = 2$ for open strings and $\beta = 1$ for closed strings. These lead to

$$\dot{X}\cdot X' = 0 \qquad \dot{X}^2 + c^2X'^2 = 0. \tag{1}$$

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{c^2}{2\pi\alpha'}X^{\mu\prime}, \tag{2}$$

$$(\partial_\tau^2 - c^2\partial_\sigma^2)X^\mu = 0. \tag{3}$$

• We will later focus on light cone gauge: $n_\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots)$. Introducing n^μ obscures the relativistic invariance in spacetime. Why would we want to do that? Well we wouldn't, except that it happens to have some other benefits once we quantize the theory. It gives a way to determine the spectrum without having to introduce unphysical states. There is a covariant approach, but it requires introducing unphysical states ("ghosts") and then ensuring that they are projected out of the physical spectrum – doing this requires sophisticated theory which is only taught at the advanced graduate student level, so we'll stick with the simpler (and in the end physically equivalent) light-cone gauge description.

• The general solution of the linear equations (3) is a superposition of Fourier modes

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0}^{\infty} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma,$$

where $\alpha_{-n}^\mu \equiv \alpha_n^{\mu *}$ (to make X^μ real) and it's also convenient to define $\alpha_0^\mu \equiv \sqrt{2\alpha'} p^\mu$. Then

$$\dot{X}^\mu \pm X^{\mu'} = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau\pm\sigma)}.$$

In light cone gauge take $n_\mu = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots)$. Then $n \cdot X = X^+$ and $n \cdot p = p^+$, so our constraint gives $X^+ = \beta\alpha' p^+ \tau$ and $p^+ = 2\pi \mathcal{P}^{\tau +}/\beta$ (again, $\beta = 2$ for open strings and $\beta = 1$ for closed strings. Also note, $X'^+ = 0$ and $\dot{X}^+ = \beta\alpha' p^+$); of course, p^+ is a constant of the motion. Since the constraints give $(\dot{X} \pm X')^2 = -2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2 = 0$, we can write this as $\partial_\tau X^- \pm \partial_\sigma X^- = \frac{1}{\beta\alpha'} \frac{1}{2p^+} (\dot{X}^I \pm X^{I'})^2$, where I are the transverse directions. This leads to

$$\sqrt{2\alpha'} \alpha_n^- = \frac{1}{p^+} L_n^\perp, \qquad L_n^\perp = \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^I \alpha_m^I.$$

This means that there is no dynamics in X^-, other than the zero mode. For $n = 0$, using $\alpha_0^- = \sqrt{2\alpha'} p^-$ get $2\alpha' p^+ p^- = L_0^\perp$. Light cone gauge allows us to make $\dot{X}^+$ a constant, and to solve for the derivatives of X^- (without having to take a square root). Finally, note that the string has

$$M^2 = -p^2 = 2p^+ p^- - P^I p^I = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_n^{I*} \alpha_n^I.$$

See that all classical states have $M^2 \geq 0$.

• Consider classical scalar field theory, with $S = \int d^D x(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2)$. The EOM is the Klein-Gordon equation

$$(\partial^2 - m^2)\phi = 0, \qquad \partial^2 \equiv -\frac{\partial^2}{\partial t^2} + \nabla^2$$

The Hamiltonian is $H = \int d^{D-1}x(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)$, where $\Pi = \partial\mathcal{L}/\partial(\partial_0\phi) = \partial_0\phi$. Take e.g. $D = 1$ and get SHO with $q \to \phi$ and $m \to 1$ and $\omega \to m$.

Classical plane wave solutions: $\phi(t, \vec{x}) = ae^{-iEt+i\vec{p}\cdot\vec{x}} + c.c.$, where $E = E_p = \sqrt{\vec{p}^2 + m^2}$, and the $+c.c.$ is to make ϕ real. Letting $\phi(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x}\phi(p)$, the reality condition is $\phi(p)^* = \phi(-p)$ and the EOM is $(p^2 + m^2)\phi(p) = 0$.

• Now consider light cone gauge coordinates. Replace $\partial^2 \to -2\partial_+\partial_- + \partial_I\partial_I$ and Fourier transform

$$\phi(x^+, x^-, \vec{x}_T) = \int \frac{dp^+}{2\pi} \int \frac{d^{D-2}\vec{p}_T}{(2\pi)^{D-2}} e^{-ix^- p^+ + i\vec{x}_T\cdot\vec{p}_T}\phi(x^+, p^+, \vec{p}_T).$$

Then the EOM becomes

$$(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^I p^I + m^2))\phi(x^+, p^+, \vec{p}_T) = 0.$$

Looks like the non-relativistic Schrodinger equation, with x_+ playing the role of time and p^+ playing the role of mass, even though it is secretly relativistic.

• Let's quantize! Replace ϕ with an operator. Consider

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}} \frac{1}{\sqrt{2E_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)e^{-ip\cdot x}).$$

If we're in a spatial box, then $p_i L_i = 2\pi n_i$. Compute the energy to find

$$H = \sum_{\vec{p}>0}(\frac{1}{2E_p}\dot{a}^\dagger\dot{a}(t) + \tfrac{1}{2}E_p a^\dagger a) = \sum_{\vec{p}} E_p a^\dagger_{\vec{p}} a_{\vec{p}}.$$

where the EOM were used in the last step: $a_{\vec{p}}(t) = a_{\vec{p}}e^{-iE_p t} + a^\dagger_{-\vec{p}}e^{iE_p t}$. Also,

$$\vec{P} = \sum_{\vec{p}} \vec{p}a^\dagger_{\vec{p}}a_{\vec{p}}.$$

As expected, H and $\vec{P}$ are independent of t. We quantize this as a (complex) SHO for each value of $\vec{p}$:

$$[a_p, a^\dagger_k] = \delta_{p,k}, \qquad [a_p, a_k] = [a^\dagger_p, a^\dagger_k] = 0.$$

and interpret the above H and $\vec{P}$ has saying that $a^\dagger_{\pm\vec{p}}$ is a creation operator, creating a state with energy $E_p = \sqrt{\vec{p}^2 + m^2}$ and spatial momentum $\vec{p}$ from the vacuum $|\Omega\rangle$. (Note that we dropped the $2 \cdot \frac{1}{2} E_p$ groundstate energy contribution, for no good reason. This is the road to the unresolved cosmological constant problem, so we won't go there.)

• Now consider the Maxwell field A^μ and quantize $\to$ photons. In the vacuum, setting $j^\mu = 0$, we have $\partial_\mu F^{\nu\mu} = 0$, which implies $\partial^2 A^\mu - \partial^\mu(\partial \cdot A) = 0$. Massless. Fourier transform to $A^\mu(p)$, with $A^\mu(-p) = A^\mu(p)^*$, and get $(p^2\eta^{\mu\nu} - p^\mu p^\nu)A_\nu(p) = 0$. Gauge invariance $\delta A_\mu(p) = ip_\mu \epsilon(p)$. In light cone gauge, since $p^+ \neq 0$, can set $A^+(p) = 0$. Then get $A^- = (p^I A^I)/p^+$, i.e. A^- is not an independent d.o.f., but rather constrained, and the Maxewell EOM gives $p^2 A^\mu(p) = 0$. For $p^2 \neq 0$, require $A^\mu(p) = 0$, and for $p^2 = 0$ get that there are $D - 2$ physical transverse d.o.f., the $A^I(p)$. The one-photon states are

$$\sum_{I=2}^{D-1} \xi_I a^{I\dagger}_{p^+,p_T} |\omega\rangle.$$

• Gravitational light cone gauge conditions: $h^{++} = h^{+-} = h^{+I} = 0$. Other light cone components are constrained. So physical d.o.f. are specified by a traceless symmetric matrix h^{IJ} in the $D - 2$ transverse directions. So $\frac{1}{2}D(D-3)$ d.o.f..