4/25/11 Lecture 9 outline

• Last time, geodesic equation: a particle in a general spacetime metric moves to extremize

$$\Delta\tau = \int d\tau = \int d\lambda \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}/c^2}. \qquad (1)$$

This is the definition of a geodesic. Gives

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0,$$

$$\Gamma^\nu_{\mu\sigma} = \tfrac{1}{2}g^{\rho\nu}(\partial_\mu g_{\rho\sigma} + \partial_\sigma g_{\rho\mu} - \partial_\rho g_{\mu\sigma})$$

So the geodesic equation is

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0.$$

When we try to solve the geodesic equations, it's useful to use integrals of the motion. While it's perhaps not yet obvious (we'll explain it more soon), one integral of the motion is

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \text{constant}.$$

For massive particles we can take $\lambda = \tau$ and the constant is -1, $u_\mu u^\mu = -1$. For photons, the constant is 0, showing that photons move on the light cone.

When the metric has a spatial symmetry isometry, then there is another integral of the motion, $\xi \cdot u = 0$ where ξ is any Killing vector isometry of the space.

• Example, FRW metric $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$. Consider extremizing $\int d\tau(-(\frac{dt}{d\tau})^2 + a^2(t)\frac{dx^i}{d\tau}\frac{dx^i}{d\tau})$, (it gives the same EOM as the $\surd$). Vary δt and $\delta\vec{x}$ to get the four EOM equations:

$$\frac{d^2x^0}{d\tau^2} + a\dot{a}\frac{dx^i}{d\tau}\frac{dx_i}{d\tau} = 0.$$

$$\frac{d^2x^i}{d\tau^2} + \frac{2\dot{a}}{a}\frac{dx^i}{d\tau}\frac{dx^0}{d\tau} = 0.$$

So we see that $\Gamma^0_{ij} = a\dot{a}\delta_{ij}$ and $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_{ij}$ are the non-zero terms (which also follows from plugging the metric into the formula above).

Consider e.g. the null paths of photons moving along the x axis, $(t(\lambda), x(\lambda), 0, 0)$ with $\frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}$. Combine this with the geodesic equation for $x^0 = t$ to get

$$\frac{d^2t}{d\lambda}^2 + \frac{\dot{a}}{a}(\frac{dt}{d\lambda})^2 = 0.$$

The solution is $\frac{dt}{d\lambda} = \omega_0/a$. This is good. The 4-velocity of some observer is u^μ with $u_\mu u^\mu = -1$ and in their rest frame $u^0 = 1/\sqrt{-g_{00}}$ and the energy of a photon is $E = -p_\mu u^\mu = -g_{00} u^0 \frac{dx^0}{d\lambda}$. So the energy measured by this comoving observer at fixed spatial coordinates is $-\sqrt{g_{00}} \frac{dx^0}{d\lambda} = \omega_0/a \sim 1/a$. This is the cosmological redshift.

• Weak field approximation: take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, and we'll work only to lowest order in h, dropping terms quadratic and higher in $h_{\mu\nu}$, e.g. $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. We can think of $h_{\mu\nu}$ as a symmetric tensor field propagating in Minkowski space. Let's consider the geodesic equation for a massive particle that's moving slowly, so $dx^i/d\tau \ll dt/d\tau$. We'll also suppose that the field is static, so $\partial_0 h_{\mu\nu} \approx 0$. In these approximations, the geodesic equation's only contribution from the connection is from $\Gamma^\mu_{00} \approx -\frac{1}{2}\partial^\mu h_{00}$ and the geodesic equation becomes

$$\frac{d^2 x^\mu}{d\tau^2} \approx -\frac{1}{2}\partial^\mu h_{00} \left(\frac{dt}{d\tau}\right)^2,$$

which gives, dividing by $dt/d\tau$s,

$$\frac{d^2 x^i}{dt^2} \approx \frac{1}{2}\partial^i h_{00}.$$

Using $h_{00} = -2\Phi$, this indeed reduces to the non-relativistic expression $\vec{a} = -\vec{\nabla}\Phi$.

• Next topic, the Schwarzschild metric, which is the basic, spherically symmetric solution of the Einstein equations of GR (which we'll meet later): $ds^2 = -(1-2GM/rc^2)dt^2 + (1-2GM/rc^2)^{-1}dr^2 + r^2 d\Omega^2$. This is the metric e.g. outside of the sun or any spherically symmetric, uncharged, un-rotating mass distribution. It's immediately apparent that something bizarre happens for $r \leq R_* = 2GM/c^2$. For ordinary objects like the sun, earth, etc $R_* \ll R_{object}$, and the above metric doesn't apply inside the object. (It's modified, analogous to how we use Gauss' law to find the electric field inside a charge distribution.) E.g. R_* for the sun is about 2.95km, way inside the sun. If $R_{object} \leq R_*$, the object is a black hole.

In GR we can use $G = c = 1$ units, measuring mass in meters. In these units $M_{sun} \approx 1.47km$ and $M_{earth} \approx 0.44cm$.