4/18/11 Lecture 7 outline

•Last time, geometric vectors $V = V^\mu e_{(\mu)}$, where V is a geometric object, invariant under coordinate transformations, but components V^μ and basis vectors $e_{(\mu)}$ do change. Also introduced dual vectors (1-forms), $\omega = \omega_\mu \theta^{(\mu)}$, which transform like e.g. $\omega_{\mu'} = \Lambda^\nu{}_{\mu'}\omega_\nu$ and $\theta^{(\mu')} = \Lambda^{\mu'}{}_\nu \theta^{(\nu)}$.

Geometric vectors have a basepoint p, which can be any point in the geometry, and the vector lies in the tangent space T_p. E.g. along $x^\mu(\lambda)$ a function has $df/d\lambda = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}$, so write

$$\frac{d}{d\lambda}|_p = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x_\mu}$$

and think of this as an example of $V = V^\mu e_{(\mu)}$, with $V = \frac{d}{d\lambda}$, $V^\mu = \frac{dx^\mu}{d\lambda}$, and $e_{(\mu)} = \frac{\partial}{\partial x^\mu}$.

For dual vectors we write e.g. $df = \frac{\partial f}{\partial x^\mu}dx^\mu$, and think of this as $\omega = \omega_\mu \theta^{(\mu)}$ with $\theta^{(\mu)} = dx^\mu$ the basis for 1-forms (cotangent space).

Again, all forces have associated local (gauge) symmetries, e.g. $SU(3) \times SU(2) \times U(1)$. Those are internal symmetries. GR is based on the symmetry principle of general coordinate invariance (or covariance): one can do local coordinate changes $x^\mu \to x^{\mu'}(x)$. Lorentz transformations are merely a special case, where it's a linear transformation with special matrices(rotations + boosts). It'll be crucial to understand how vectors transform under these changes.

Geometric vectors or dual vectors are unchanged. Their components and basis vectors do change, oppositely, generalizing what we've seen for Lorentz transformations. The general change can be nicely understood in coordinate basis, simply using the chain rule:

$$\frac{\partial x^\mu}{\partial \lambda} = \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{\partial x^\nu}{\partial \lambda}, \qquad \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\nu} \to V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}V^\nu, \qquad e_{(\mu')} = \frac{\partial x^\nu}{\partial x^{\mu'}}e_{(\nu)},$$

and note that $V^\mu e_{(\mu)} = V^{\mu'}e_{(\mu')}$ is invariant, good.

Likewise, considering $df = \frac{\partial f}{\partial x^\mu}dx^\mu$ as an example of a dual vector, $\omega = \omega_\mu \theta^{(\mu)}$ shows that their components and basis vectors transform as $\omega_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}}\omega_\nu$ and $\theta^{(\mu')} = \frac{\partial x^{\mu'}}{\partial x^\nu}\theta^{(\nu)}$.

• Illustrate this for changes between rectangular and polar or spherical coordinates.

Now consider $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The LHS is geometric, so it's unchanged by coordinate transformations, but the components $g_{\mu\nu}$ transform, oppositely from $dx^\mu dx^\nu$, $g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g_{\mu\nu}$.

Example: $dS^2 = d\vec{x}\cdot d\vec{x} = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$, so we know $g_{\phi\phi} = r^2\sin^2\theta$, and we can also get that from $g_{\phi\phi} = \frac{\partial x^\mu}{\partial \phi}\frac{\partial x^\nu}{\partial \phi}g_{\mu\nu} = \sum_{i=1}^3 \left(\frac{\partial x^i}{\partial \phi}\right)^2$.

Likewise for other tensors from special relativity, like $F^{\mu\nu}$ or $T^{\mu\nu}$, e.g. $T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} T^{\mu\nu}$. You can always get it right by paying attention to the indices and whether they're raised or lowered.

The special case of Lorentz transformations is just distinguished by being linear and mapping the flat metric $\eta_{\mu\nu}$ to itself.

• Not everything is a tensor. Consider a vector A_μ (it could be e.g. $A^\mu = (\phi, \vec{A})$ of E&M). Then $\partial_\mu A_\nu$ is not a tensor. On the other hand, $\partial_{[\mu} A_{\nu]}$ is a tensor. More later.

• The proper time measured by an observer is general coordinate invariant. It is given by $d\tau^2 = -ds^2/c^2 = -g_{\mu\nu} dx^\mu dx^\nu / c^2$. This fits with what we said before, where the effect of a gravitational potential was included via $g_{00} \approx 1 + 2\Phi(x)/c^2$ to describe clocks running at different rates in a graviational potential.

• Equivalence principle $\rightarrow$ a particle in a general spacetime metric has

$$S/mc^2 = -\Delta\tau = -\int d\tau = -\int d\lambda \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} / c^2}. \tag{1}$$

For any metric, the principle of least action means that the particle moves on the path of maximal proper time! This is the definition of a geodesic. The geodesic equation can be derived from the E.L. equations for proper time.

What about photons. Even though they have $m = 0$, we can still find how they move by the relativistic version of Fermat's principle: they too move so as to extremize their proper time $\Delta\tau$. So photons also follow geodesics! Their equations of motion also follow from the EL equations for the RHS of (1).