4/13/11 Lecture 6 outline

• More nice examples from Hartle Ch 7.

Wormhole spacetime (illustrating embedding). Consider $ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta d\phi^2)$ on slice t =const. and $\theta = \pi/2$: $d\Sigma^2 = dr^2 + (b^2 + r^2 d\phi^2)$. Get it from flat cylindrical coordinates $dS^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2$ via $z = z(r)$ and $\rho = \rho(r)$, and $\psi = \phi$, which gives $d\Sigma^2 = (z'(r)^2 + \rho'(r)^2)dr^2 + \rho^2 d\phi^2$, which gives the metric above for $\rho^2 = r^2 + b^2$ and $\rho(z) = b\cosh(z/b)$. Plot in (z, ρ) plane and see need negative r to get negative z. Embedding (r, ϕ) as a 2d surface in flat 3d space gives the wormhole picture with two asymptotically flat regions connected by throat of length $2\pi b$.

• Friedman Robertson Walker metric (more later!): $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$ solves the Einstein equations (which we'll see later) for a perfect fluid, taking $p = w\rho$, with scale factor $a(t) = t^q$ for $q = 2/3(1+w)$. Cosmology, with $t = 0$ the big bang. Light cones have $\frac{dx}{dt} = \pm t^{-q}$.

• Geometric quantities from metrics. Taking $ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$, find area elements e.g. $dA = \sqrt{g_{11}g_{22}}dx^1 dx^2$, 3-volume elements $d\mathcal{V} = \sqrt{g_{11}g_{22}g_{33}}dx^1 dx^2 dx^3$, and 4-volume element $dv = \sqrt{-g_{00}g_{11}g_{22}g_{33}}d^4x$. Examples, e.g. spherical coordinates.

• Hypersurfaces, e.g. $x^0 = h(x^1, x^2, x^3)$ is a spacelike slice. Tangents are $\mathbf{t}$ and normal is $\mathbf{n}$, with $n \cdot t = 0$, and the surface is spacelike as long as $n \cdot n < 0$. E.g. Lorentz Hyperboloid, $-t^2 + r^2 = a^2$. Write $t = a\cosh\chi$ and $r = a\sinh\chi$, so the tangent is $t^\mu = (a\sinh\chi, a\cosh\chi, 0, 0)$ and normal is $n^\mu = (\cosh\chi, \sinh\chi, 0, 0)$, with $n \cdot n = -1$.

• Back to vectors, $V = V^\mu e_{(\mu)}$, where V is a geometric object, invariant under coordinate transformations, but components V^μ and basis vectors $e_{(\mu)}$ do change, e.g. under $x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$, then $V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$, and $e_{(\mu')} = \Lambda^\nu{}_{\mu'} e_{(\nu)}$. Also introduce dual vectors (1-forms), $\omega = \omega_\mu \theta^{(\mu)}$, which transform like e.g. $\omega_{\mu'} = \Lambda^\nu{}_{\mu'}\omega_\nu$ and $\theta^{(\mu')} = \Lambda^{\mu'}{}_\nu \theta^{(\nu)}$.

Geometric vectors have a basepoint p, which can be any point in the geometry, and the vector lies in the tangent space T_p. E.g. along $x^\mu(\lambda)$ a function has $df/d\lambda = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}$, so write

$$\frac{d}{d\lambda}|_p = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x_\mu}$$

and think of this as an example of $V = V^\mu e_{(\mu)}$, with $V = \frac{d}{d\lambda}$, $V^\mu = \frac{dx^\mu}{d\lambda}$, and $e_{(\mu)} = \frac{\partial}{\partial x^\mu}$.

For dual vectors we write e.g. $df = \frac{\partial f}{\partial x^\mu}dx^\mu$, and think of this as $\omega = \omega_\mu \theta^{(\mu)}$ with $\theta^{(\mu)} = dx^\mu$ the basis for 1-forms (cotangent space).

• Again, all forces have associated local (gauge) symmetries, e.g. $SU(3) \times SU(2) \times U(1)$. Those are internal symmetries. GR is based on the symmetry principle of general coordinate invariance (or covariance): one can do local coordinate changes $x^\mu \to x^{\mu'}(x)$. Lorentz transformations are merely a special case, where it's a linear transformation with special matrices(rotations + boosts). It'll be crucial to understand how vectors transform under these changes.

Geometric vectors or dual vectors are unchanged. Their components and basis vectors do change, oppositely, generalizing what we've seen for Lorentz transformations. The general change can be nicely understood in coordinate basis, simply using the chain rule:

$$\frac{\partial x^\mu}{\partial \lambda} = \frac{\partial x^{\mu'}}{\partial x^\nu}\frac{\partial x^\nu}{\partial \lambda}, \qquad \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\nu} \to V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}V^\nu, \qquad e_{(\mu')} = \frac{\partial x^\nu}{\partial x^{\mu'}}e_{(\nu)},$$

and note that $V^\mu e_{(\mu)} = V^{\mu'} e_{(\mu')}$ is invariant, good.

Likewise, considering $df = \frac{\partial f}{\partial x^\mu}dx^\mu$ as an example of a dual vector, $\omega = \omega_\mu \theta^{(\mu)}$ shows that their components and basis vectors transform as $\omega_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}}\omega_\nu$ and $\theta^{(\mu')} = \frac{\partial x^{\mu'}}{\partial x^\nu}\theta^{(\nu)}$.