4/11/11 Lecture 5 outline

⋆ For today's lecture: Hartle chapter 7.

• Last time: we saw that $d\tau(1-\Phi/c^2) \equiv dt$ coincides for the two observers. So we write $\Delta\tau_B \approx (1+\Phi_B/c^2)\Delta t$ and $\Delta\tau_A \approx (1+\Phi_A/c^2)\Delta t$. In gravity, the time interval Δt is artificial but still useful; it's called the "coordinate" time interval. We'll distinguish between "coordinate" quantities vs. physical quantities, like proper time.

Draw a (x,t) diagram of the events of emission of light pulses at x_A and their detection at x_B. The light pulses are separated by coordinate separation Δt for both. But the proper time intervals are different at the two places. This will be interpreted as coming from a spacetime metric that differs from $\eta_{\mu\nu}$. We have $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\mu$, where $g_{\mu\nu}(x)$ is the spacetime metric.

In particular, for the above, writing proper time as $cd\tau = \sqrt{-ds^2}$, we have

$$ds^2(x) \approx -(1+2\Phi(x)/c^2)(cdt^2) + f(x)d\vec{x}^2, \tag{1}$$

where all we know at this point is that $f(x) \approx 1$ to leading order in $1/c^2$ (we'll see that GR gives $f(x) \approx 1 - 2\Phi(x)/c^2$ as the leading correction in $1/c^2$, but we don't need that yet). So we have $g_{\mu\nu} \approx \eta_{\mu\nu} + (2\Phi/c^2)\delta_{\mu 0}\delta_{\nu 0}$. The interval (1) is our first glimpse of connecting gravity to spacetime curvature.

Look at a map of the earth, and the distance between Paris and Montreal vs that between Lagos and Bogota. The flat-earth theory, that the latter only seems shorter, because rulers shrink closer to the North and South "poles". Better to think in terms of the curved geometry of the globe vs the flat geometry of maps.

• Let's see how the above works. We saw before that a free particle has $S = -mc^2\int d\tau$. Using the above, we have

$$S \approx -mc^2 \int dt\sqrt{1+2\Phi(x)/c^2 - \vec{v}^2/c^2} \approx \int dt(-mc^2 + \tfrac{1}{2}m\vec{v}^2 - m\Phi(x))$$

where $f(x)$ drops out to our order in $1/c^2$. The last expression indeed reproduces the non-relativistic action for a massive particle in the gravitational potential $V(x) = m\Phi(x)$ – it works!

• An appetizer: write $\Phi = -GM/r$, and then note that we have $g_{00} \approx 1 - 2GM/r$. Though we've dropped terms of higher order in Φ/c^2, we'll see later that this is indeed the exact g_{00} for a spherical mass (provided that it has no charge or angular momentum).

It looks like something funny happens at $r_* = 2GM$: that is the horizon of a black hole. More to follow.

$\star$ Next topic: a bit on the mathematics of curved spacetime, see Hartle ch. 7. Contrast approaches of Hartle vs standard GR texts like Carroll. Analogy with E&M. We could defer discussion about how $\vec{E}$ and $\vec{B}$ are actually created (i.e. defer Maxwell's equations), just treat them as fixed quantities, and first study how to solve the Lorentz force equation for small charges in those external fields. Likewise in gravity, we can defer discussion of how gravity creates curved spacetime (i.e. defer Einstein's equations) and first study how small masses move in an external background metric, by the "geodesic equation." This is the approach that Hartle takes. The advantage is that we can defer some of the math discussion. We'll try this approach here this time. So we'll cover a minimal amount of math now, to get to the physics of geodesic motion. Later, we'll cover more math and get to Einstein's equations.

• Flat spacetime in various coordinate systems: $ds^2 = -dt^2 + d\vec{x}^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 = -dx^2 + dy^2 + y^2 dz^2 + y^2 \sin^2 z dt^2$ (the last one is just trying to fool you with some funky name choices). Consider a plane with $dS^2 = dr^2 + r^2 d\theta^2$ and $r' = a^2/r$ to get $dS'^2 = (a^4/r'^4)(dr'^2 + r'^2 d\phi^2)$. Apparent singularity at $r' = 0$, but it's fake.

• Penrose diagram of flat space: $u \equiv t - r$, $v \equiv t + r$, $ds^2 = -dudv + \frac{1}{4}(u-v)^2(d\theta^2 + \sin^2\theta d\phi^2)$. Now replace $u' = \tan^{-1} u$ and $v' = \tan^{-1} v$. Now write $u' = \frac{1}{2}(t' - r')$ and $v' = \frac{1}{2}(t' + r')$. The entire (t, r) plane is mapped to the finite triangular region $r' > 0$, $v' < \pi/2$, $u' > -\pi/2$. This is an example of a Penrose diagram, which have the property that the global structure is exhibited in a finite picture by a conformal transformation, so light still travels on a 45° line, with past and future timelike directions inside that line, and spacelike directions outside it. The boundary $u' = -\pi/2$ is $\mathcal{I}_-$ past null infinity (where all light rays start) and the boundary $v = \pi/2$ is $\mathcal{I}_+$, future null infinity (where all light rays end).

• Metric $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$. We'll study how things are affected by coordinate change. Mathematical statement: for any metric $g_{\mu\nu}$, we can introduce locally, at any point p, a coordinate system x'_p such that the geometry there looks locally flat:

$$g'_{\mu\nu}(x'_p) = \eta_{\mu\nu} \qquad \text{and} \qquad \frac{\partial g'_{\mu\nu}}{\partial x'^\rho}|_{x=x_p} = 0.$$

This mathematical statement will connect with the equivalence principle: locally any spacetime is indistinguishable from the flat spacetime of special relativity. Globally is a different story. Flat maps of small regions of the earth are fine, but globally we need the globe.

Example: sphere at North Pole: $ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2$, near $\theta = 0$ use $x = a\theta\cos\phi$, $y = a\theta\sin\phi$, find $ds^2 = (1 - 2y^2/3a^2)dx^2 + (1 - 2x^2/3a^2)dy^2 + 4xydxdy/3a^2 + \mathcal{O}(\xi^3)$. This is an example of Gaussian normal coordinates.

• Preview **General covariance**: the laws of physics are invariant under any coordinate change $x^\mu \to x^{\mu'}(x^\mu)$. **Equivalence principle:** locally at any point there exists a choice of freely falling coordinates $x^{\widehat{\mu}}$ such that massive timelike objects have $d^2x^{\widehat{\mu}}/d\tau^2 = 0$ and $d\tau^2 = -\eta_{\widehat{\mu\nu}}dx^{\widehat{\mu}}dx^{\widehat{\nu}}$. Massless null or lightlike objects have $d^2x^{\widehat{\mu}}/d\sigma^2 = 0$ and $0 = \eta_{\widehat{\mu\nu}}dx^{\widehat{\mu}}dx^{\widehat{\nu}}$.

• More nice examples from Hartle Ch 7.

Worldline $X^\mu(\tau)$, e.g. consider metric $ds^2 = -X^2dT^2 + dX^2$ and worldline $X(T) = A\cosh T$, get $d\tau^2 = A^2(\cosh^2 T - \sinh^2 T)dT^2 = A^2dT^2$, so write $X^\mu(\tau)$ as $(T(\tau), X(\tau) = (\tau/A, A\cosh(\tau/A))$.

Warp drive spacetime: Hans Solo's craft travels on $x_s(t)$ (and $y = z = 0$) and locally modifies the spacetime metric to $ds^2 = -dt^2 + (dx - V_s(t)f(r_s)dt)^2 + dy^2 + dz^2$, where $V_s(t) = dx_s(t)/dt$ and $f(r_s)$ is a function that's 1 near the ship $r_s = 0$ and zero away, where r_s is the spatial distance away from the ship. Light cone is $ds^2 = 0$, so $dx/dt = \pm 1 + V_s(t)f(r_s)$, gets tipped forward along the direction of motion of the ship. Can connect stations with $\Delta t < \Delta x$ by having $V_s(t) > 1$ in some regions, but still have ship always stay within its own local light cone.

Such spacetimes need to use negative energy densities, excluded by classical physics (though not by quantum effects).

Wormhole spacetime (illustrating embedding). Consider $ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta d\phi^2)$ on slice $t =$const. and $\theta = \pi/2$: $d\Sigma^2 = dr^2 + (b^2 + r^2 d\phi^2)$. Get it from flat cylindrical coordinates $dS^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2$ via $z = z(r)$ and $\rho = \rho(r)$, and $\psi = \phi$, which gives $d\Sigma^2 = (z'(r)^2 + \rho'(r)^2)dr^2 + \rho^2 d\phi^2$, which gives the metric above for $\rho^2 = r^2 + b^2$ and $\rho(z) = b\cosh(z/b)$. Plot in (z, ρ) plane and see need negative r to get negative z. Embedding (r, ϕ) as a 2d surface in flat 3d space gives the wormhole picture with two asymptotically flat regions connected by throat of length $2\pi b$.

• Taking $ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$, find area elements e.g. $dA = \sqrt{g_{11}g_{22}}dx^1dx^2$, 3-volume elements $d\mathcal{V} = \sqrt{g_{11}g_{22}g_{33}}dx^1dx^2dx^3$, and 4-volume element $dv = \sqrt{-g_{00}g_{11}g_{22}g_{33}}d^4x$.

• Back to vectors, $V = V^\mu e_{(\mu)}$, where V is a geometric object, invariant under coordinate transformations, but components V^μ and basis vectors $e_{(\mu)}$ do change, e.g. under $x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$, then $V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$, and $e_{(\mu')} = \Lambda^\nu{}_{\mu'} e_{(\nu)}$. Also introduce dual vectors (1-forms), $\omega = \omega_\mu \theta^{(\mu)}$, which transform like e.g. $\omega_{\mu'} = \Lambda^\nu{}_{\mu'}\omega_\nu$ and $\theta^{(\mu')} = \Lambda^{\mu'}{}_\nu \theta^{(\nu)}$.

Geometric vectors have a basepoint p, which can be any point in the geometry, and the vector lies in the tangent space T_p. E.g. along $x^\mu(\lambda)$ a function has $df/d\lambda = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}$, so write

$$\frac{d}{d\lambda}|_p = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x_\mu}$$

and think of this as an example of $V = V^\mu e_{(\mu)}$, with $V = \frac{d}{d\lambda}$, $V^\mu = \frac{dx^\mu}{d\lambda}$, and $e_{(\mu)} = \frac{\partial}{\partial x^\mu}$.

For dual vectors we write e.g. $df = \frac{\partial f}{\partial x^\mu} dx^\mu$, and think of this as $\omega = \omega_\mu \theta^{(\mu)}$ with $\theta^{(\mu)} = dx^\mu$ the basis for 1-forms (cotangent space).

• Hypersurfaces, e.g. $x^0 = h(x^1, x^2, x^3)$ is a spacelike slice. Tangents are **t** and normal is **n**, with $n \cdot t = 0$, and the surface is spacelike as long as $n \cdot n < 0$. E.g. Lorentz Hyperboloid, $-t^2 + r^2 = a^2$. Write $t = a\cosh\chi$ and $r = a\sinh\chi$, so the tangent is $t^\mu = (a\sinh\chi, a\cosh\chi, 0, 0)$ and normal is $n^\mu = (\cosh\chi, \sinh\chi, 0, 0)$, with $n \cdot n = -1$.