4/6/11 Lecture 4 outline

⋆ For today's lecture: Hartle chapter 6.

• Last time: Equivalence principle

WEP: $m_i = m_g$. LHS: enters in $\vec{F} = m_i\vec{a}$, and RHS enters in $\vec{F}_g = -m_g\nabla\Phi$, so $\vec{a} = -\nabla\Phi$. Can replace gravity with $\langle a\rangle_{obs} = -\vec{a} = +\nabla\Phi$.

• Aside on history: $m_i = m_g$ was tested in the 6th century, in 1586, in 1610 (Galileo), 1680 (Newton), 1832, by dropping balls, rolling balls down inclines, measuring periods of pendulums with same length. Eotvos in 1908 gave a quantitative measurement, $m_i = m_g$ to one part in 10^9, using his Eotvos balance (which he developed to measure density variations underground, masses of buildings etc.). Recent Eot-Wash group brought the bound to one part in 10^{13}.

Can't distinguish between gravity and acceleration. Motion of freely falling particles locally same in gravity field vs a uniformly accelerating frame. Eotvos experiment.

EEP: In small regions laws of physics reduce to special relativity, freely falling small observers can't detect gravity by any local experiment. E.g. binding energy of hydrogen atom contributes s.t. $m_i = m_g$ is preserved.

SEP: **All** laws of physics are such that gravity can't be detected by local, free falling observer's experiments. Tom Murphy's lunar ranging experiment can look for e.g. how the earth and moon which, together with their gravitational self-energy, are together in free-fall around the sun.

• Theorists age more quickly at UCSD! Clocks run slower where the gravitational potential is lower. Small effect, but directly measurable with atomic clocks: a clock one meter above the ground will run faster than one on the ground. The GPS system is sufficiently precise that it needs to acount for this.

Understand the effect using the equivalence principle, replacing gravity with an accelerating rocket. B on bottom of rocket has $z_B(t) \approx \frac{1}{2}gt^2$ and A on top has $z_A(t) \approx h + \frac{1}{2}gt^2$. A emits a light pulse at $t = 0$ and B detects it at $t = t_1$, where $z_A(0) - z_B(t_1) = ct_1$. Then A emits another pulse at $t = \Delta\tau_A$ and B detects it at $t = t_1 + \delta\tau_B$, where $z_A(\Delta\tau_A) - z_B(t_1 + \Delta\tau_B) = c(t_1 + \Delta\tau_B - \Delta\tau_A)$. Assume $gh \ll c^2$, and keep only to first order in this small quantity. Also assume $\Delta\tau_A$ and $\Delta\tau_B$ are small, so keep only to linear order in them. Everything is kept non-relativisitic, so it doesn't matter if we measure times and lengths in A's, B's, or outside frames. Then find $t_1 \approx h/c$ and $\Delta\tau_B \approx \Delta_A(1 - gh/c^2)$.

Write $\vec{a}_{rocket} = \nabla\Phi$, where we're going to identify Φ with the gravitational potential. So $gh = \Phi_A - \Phi_B$ and we have found

$$\Delta\tau_B \approx (1 - (\Phi_A - \Phi_B)/c^2)\Delta\tau_A.$$

Let's also consider the frequency of the light in each pulse. A sends light with frequency $\omega_A = -k_\mu u_A^\mu$, then B detects it as having frequency $\omega_B = -k_\mu u_B^\mu$. Consider the first pulse and write $k^\mu = \omega_*(1,0,0,-1)$ and $u^\mu(t) \approx (1,0,0,gt/c)$, then $\omega_A \approx \omega_*$ and $\omega_B \approx \omega_*(1 + gt_1/c) \approx \omega_*(1 + gh/c^2)$. Observer B sees a blue-shift because of their increased velocity into the direction of the light source. The period of the light according to the two observers are related by the same expression that we found above for the time between pulses: $\Delta\tau_B \approx \Delta\tau_A(1 - gh/c^2)$.

Let's write it another way, for fun, the Doppler shift of the photons for a small velocity change is: $\begin{pmatrix} \omega + d\omega \\ \omega + d\omega \end{pmatrix} = \begin{pmatrix} \gamma_{dv} & -\gamma_{dv} dv \\ -\gamma_{dv} dv & \gamma_{dv} \end{pmatrix} \begin{pmatrix} \omega \\ \omega \end{pmatrix}$, which gives $d\omega/\omega = -dv/c = -adx/c^2 = -d\Phi/c^2$, and then integrating gives $\omega_B/\omega_A = e^{-(\Phi_B - \Phi_A)/c^2}$, so $\omega_A e^{\Phi_B/c^2} = \omega_A e^{\Phi_A/c^2}$, which we should only take seriously to leading order in Φ/c^2.

Time intervals between events are always shorter on B's clock than on A's. This is because B's local clock is running slower than A's, so less time elapse is measured by B than A for the same two events.

- We saw that $d\tau(1 - \Phi/c^2) \equiv dt$ coincides for the two observers. So we'll write $\Delta\tau_B \approx (1 + \Phi_B/c^2)\Delta t$ and $\Delta\tau_A \approx (1 + \Phi_A/c^2)\Delta t$. In gravity, the time interval Δt is artificial but still useful; it's called the "coordinate" time interval. We'll distinguish between "coordinate" quantities vs. physical quantities, like proper time.
- Draw a (x,t) diagram of the events of emission of light pulses at x_A and their detection at x_B. The light pulses are separated by coordinate separation Δt for both. But the proper time intervals are different at the two places. This will be interpreted as coming from a spacetime metric that differs from $\eta_{\mu\nu}$. We have $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\mu$, where $g_{\mu\nu}(x)$ is the spacetime metric.
- In particular, for the above, writing proper time as $cd\tau = \sqrt{-ds^2}$, we have

$$ds^2(x) \approx -(1 + 2\Phi(x)/c^2)(cdt^2) + f(x)d\vec{x}^2, \tag{1}$$

where all we know at this point is that $f(x) \approx 1$ to leading order in $1/c^2$ (we'll see that GR gives $f(x) \approx 1 - 2\Phi(x)/c^2$ as the leading correction in $1/c^2$, but we don't need that yet).

So we have $g_{\mu\nu} \approx \eta_{\mu\nu} + (2\Phi/c^2)\delta_{\mu 0}\delta_{\nu 0}$. The interval (1) is our first glimpse of connecting gravity to spacetime curvature.

• Look at a map of the earth, and the distance between Paris and Montreal vs that between Lagos and Bogota. The flat-earth theory, that the latter only seems shorter, because rulers shrink closer to the North and South "poles". Better to think in terms of the curved geometry of the globe vs the flat geometry of maps.

• **Next time:** Let's see how the above works. We saw before that a free particle has $S = -mc^2 \int d\tau$. Using the above, we have

$$S \approx -mc^2 \int dt \sqrt{1 + 2\Phi(x)/c^2 - \vec{v}^2/c^2} \approx \int dt(-mc^2 + \tfrac{1}{2}m\vec{v}^2 - m\Phi(x))$$

where $f(x)$ drops out to our order in $1/c^2$. The last expression indeed reproduces the non-relativistic action for a massive particle in the gravitational potential $V(x) = m\Phi(x)$ – it works!

• An appetizer: write $\Phi = -GM/r$, and then note that we have $g_{00} \approx 1 - 2GM/r$. Though we've dropped terms of higher order in Φ/c^2, we'll see later that this is indeed the exact g_{00} for a spherical mass (provided that it has no charge or angular momentum). It looks like something funny happens at $r_* = 2GM$: that is the horizon of a black hole. More to follow.