3/30/11 Lecture 2 outline

• Last time, introduced 4-vectors a^μ, and their inner product $a\cdot b \equiv a^\mu b^\nu \eta_{\mu\nu} \equiv a^\mu b_\mu$. Two inertial frames of reference are related by (taking origins to coincide) $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. The dot product is preserved as long as

$$\eta_{\rho\sigma} = \Lambda^{\mu'}_\rho \Lambda^{\nu'}_\sigma \eta_{\mu'\nu'}.$$

All Λ satisfying this form the Lorentz group. Note that all such Λ have determinant ± 1, and all those connected to the identity have determinant 1, so they have $d^4x = d^4x'$.

Examples: rotate in x, y plane $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$; boost along x axis, $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$. Consider the origin $x' = 0$ in the original frame, $x/t = v = \tanh\phi$, so $\sinh\phi = \gamma v$ and $\cosh\phi = \gamma \equiv 1/\sqrt{1 - v^2/c^2}$. Set $c = 1$ from now on.

Heartbeat in $'$ frame: dt', with $dx' = 0$, get $dt = \gamma dt'$, so seems to beat slower (likewise from $ds^2 = -dt^2 + d\vec{x}^2 = -dt'^2$.

Ruler in $'$ frame, length dx. Measure both ends simultaneously in lab, with $dt = 0$, Then $dx = dx'/\gamma$, length contracted.

Two events are timelike separated if there is a frame where they happen a the same place. In that frame, $\Delta s^2 = \Delta t'^2 \equiv \Delta\tau^2$, where $\Delta\tau$ is the "proper time" between the events. In any other frame, $\Delta t = \gamma\Delta\tau$, time dilation.

For spacelike path, $\Delta s = \int ds = \int \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$.

For timelike paths, the total proper time is $\Delta\tau = \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$. This applies even if there is acceleration. If no acceleration, can write $\Delta\tau = \int \sqrt{1 - v^2}dt$.

Consider proper time between timelike separated events A and C. For observer 1, in the frame where they're at the same place, the proper time is $\delta t = t_C - t_A$. For observer 2, who moves and comes back, the proper time length is $\Delta\tau_{AB'C} = \sqrt{1 - v^2}\Delta\tau_{ABC} < \Delta\tau_{ABC}$. Moving twin is younger when they meet again. Non-straight path has shorter proper time. In spacetime, straight path between two events has the longest proper time.

• Upper vs lower indices, canonical examples: $dx^\mu = (dt, d\vec{x})$ and $\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$, e.g. $\partial_\mu x^\nu = \delta^\nu_\mu$. Under $x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$, have $dx^{\mu'} = \Lambda^{\mu'}_\nu dx^\nu$ and $\frac{\partial}{\partial x^{\mu'}} = \Lambda^\nu_{\mu'}\frac{\partial}{\partial x^\nu}$, where $\Lambda^\nu_{\mu'}$ is the inverse to $\Lambda^{\mu'}_\nu$.

• 4-velocity, $u^\mu = dx^\mu/d\tau$, so $u^\mu u_\mu = -1$. $u^\mu = (\gamma, \gamma\vec{v})$.

• 4-acceleration $a^\mu = d^2x^\mu/d\tau^2$, satisfies $a^\mu u_\mu = 0$.

• 4-momentum $p^\mu = (E, \vec{p})$. For massive particle, $p^\mu = mu^\mu$, so $p^\mu p_\mu = -m^2$. For a massless object (e.g. photon), we still have $p^\mu = (E, \vec{p})$ as a 4-vector. Here's a way to see that $(E,\vec{p})$ always transforms as a 4-vector. For any theory, the action S must be Lorentz invariant; this ensures that the EOM behave properly under reference frame changes. Now use the fact that energy and momentum can be related to the derivative of the action w.r.t. changes of the endpoint time and position: $L(x_b)-\dot{x}_b(\partial L/\partial\dot{x}_b)=\partial S_{cl}/\partial t_b$, and $\partial L/\partial\dot{x}_b=\partial S_{cl}/\partial x_b$, so we have $p_\mu=\partial S_{cl}/\partial x^\mu$, and the RHS is clearly a 4-vector.

• $k^\mu=(\omega,\vec{k})$, so $e^{ik\cdot x}$ is invariant. On this, $k_\mu=i\partial_\mu$. Fits with QM, where $p^\mu=\hbar k^\mu$.

• Free particle action $S=\int Ldt$, $L=-mc^2\sqrt{1-v^2/c^2}$. Then $\vec{p}=\partial L/\partial v$ and $H=\vec{p}\cdot\vec{v}-L$ combine into $p^\mu=mu^\mu$. The EOM is then $du^\mu/d\tau=0$ for a free particle.

• Force $f^\mu=\frac{dp^\mu}{d\tau}=(\gamma\frac{dE}{dt},\gamma\frac{d\vec{p}}{dt})$.

• Charge / current density $J^\mu=(\rho,\vec{J})=\sum_i q_i(1,\dot{\vec{x}}_i)\delta^3(\vec{x}-\vec{x}_i(t))=\rho\frac{dx^\mu}{dx^0}$. Compare $\rho d^3x=\rho' d^3x'$ vs $d^4x=d^4x'$.

• Conservation law: $Q_{encl}=\int\rho dV$ has $\dot{Q}=-\oint_S\vec{J}\cdot d\vec{a}$, taking $\partial_\mu J^\mu=0$. This conservation law is related to a local symmetry, gauge invariance (more to follow), and E&M is a consequence of this symmetry.

• All forces are related to local symmetries. GR shows that gravity is related to local general coordinate transforms. A special case of that is translations, whose conserved Noether charge you know is energy and momentum.

• Conservation of p^μ is related to symmetry under translations in x^μ. Write energy, momentum as conserved charge, with corresponding density: the energy momentum tensor.

Energy momentum tensor, $P^\mu_{encl}=\int T^{0\mu}dV$ has $\dot{P}^\mu_{encl}=-\oint_S T^{i\mu}da^i$, where T^{00} is energy density, T^{0i} is momentum density / energy flux, and T^{ij} is pressure etc. So $\partial_\nu T^{\mu\nu}=0$ (in flat spacetime.

• The energy-momentum tensor (or stress-energy tensor) is a tensor: under coordinate transformations $x^{\mu'}=\Lambda^{\mu'}_\rho x^\rho$, get $T^{\mu'\nu'}=\Lambda^{\mu'}_\rho\Lambda^{\nu'}_\sigma T^{\rho\sigma}$.

• Examples of energy momentum tensors. For a bunch of particles, $T^{\mu\nu}=\sum_n p^\mu_n\frac{dx^\nu_n}{dt}\delta^3(\vec{x}-\vec{x}_n(t))$. Note that $dx^\nu_n/dt=(1,\vec{v})=p^\nu_n/E_n$, so $T^{\mu\nu}$ is symmetric.

For a perfect fluid, $T^{\mu\nu}=diag(\rho,p,p,p)$, in the rest frame. So $T^{\mu\nu}=(\rho+p)u^\mu u^\nu+p\eta^{\mu\nu}$. Illustrates a nice technique: find tensor expression from starting in the rest frame.

Example: $T^{\mu\nu}_{CC}=-\Lambda\eta^{\mu\nu}$, has $\rho_{CC}=-p_{CC}$.

• Back to E&M: $\frac{dp^\mu}{d\tau}=qF^{\mu\nu}u_\nu$, where $F^{0i}=E^i$ and $F^{ij}=\epsilon^{ijk}B_k$. Here $F^{\mu\nu}=-F^{\nu\mu}$. Under Lorentz transformations, $F^{\mu'\nu'}=\Lambda^{\mu'}_\rho\Lambda^{\nu'}_\sigma F^{\rho\sigma}$. Maxwell's equations are $\partial_\mu F^{\nu\mu}=J^\nu$ (which implies $\partial_\nu J^\nu=0$), and $\partial_\mu F_{\nu\lambda}$+cyclic$=0$. Solve the second via $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, with $A^\mu=(\phi,\vec{A})$. Note gauge invariance $A^\mu\to A^\mu+\partial^\mu f$. In Coulomb gauge take $\partial_\mu A^\mu=0$ and then get $\partial^2A^\nu=-J^\nu$. Plane wave solutions like $A^\mu=\epsilon^\mu(k)e^{ik\cdot x}$.

For a massive charged particle, $S = -m\int d\tau + q\int A_\mu dx^\mu$. Gives $\vec{p} = \partial L/\partial \vec{v} = \gamma m\vec{v} + q\vec{A}$, and $H = \vec{p}\cdot\vec{v} - L = \gamma m + q\phi = \sqrt{m^2 + (\vec{p} - q\vec{A})^2} + q\phi$.

• Classical field theory, e.g. for a scalar field: $S = \int d^4x \mathcal{L}(\Phi, \partial_\mu \Phi)$, with EL equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} - \frac{\partial L}{\partial \Phi} = 0.$$

Example, $\mathcal{L} = -\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - V(\Phi)$, get EL equations $(\partial_t^2 - \nabla^2)\Phi + \frac{dV}{d\Phi} = 0$.

In E&M, we have instead a classical field theory for $A^\mu(x)$, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$. Varying w.r.t. A_μ, the EL equations give the Maxwell equations $\partial_\mu F^{\nu\mu} = J^\nu$.

The energy-momentum tensor is the conserved Noether current related to space-time translation invariance. Get

$$T^{\mu\nu}_{E\&M} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}.$$