5/25/11 Lecture 18 outline

• Continue from last time.

$$S[g,X] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_{\text{everything else}}[\eta, X, \partial_\mu X]\big|_{\eta\to g, \partial\to\nabla}. \qquad (1)$$

(Last time we left the coefficient of the gravity term as α/G, and left it that we'd determine the coefficient by checking agreement with the Newtonian limit. Let's now just cut to the chase and write the answer, and check that it's right.)

The variation of (1) with $g_{\mu\nu} \to g_{\mu\nu} + \delta g^{\mu\nu}$ gives

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R_{\mu\nu}g^{\mu\nu} + 8\pi G T_{\mu\nu} \qquad (2)$$

where we used the relation discussed last time,

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_{everything\ else}}{\delta g^{\mu\nu}}. \qquad (3)$$

A way to relate this to the usual notion is to consider translations $x^{\mu'} = x^\mu + a^\mu$ and then, linearizing in small a^μ, get $\delta g^{\mu\nu} \approx a^{\mu;\nu} + a^{\nu;\mu}$, where the ; means covariant derivative. Then get $\delta S = \int d^4x T_{\mu\nu}a^{\mu;\nu}$. Note this shows $T^{;\nu}_{\mu\nu} = 0$, covariant energy-momentum conservation. For a macroscopic body, get $T_{\mu\nu} = (p+\rho)u_\mu u_\nu + pg_{\mu\nu}$.

Now use $g = e^{-\mathrm{Tr}\ln g^{\mu\nu}}$ to get $\delta g = -gg_{\mu\nu}\delta g^{\mu\nu}$, so $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. Also, get that $\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu} - (\lambda \leftrightarrow \nu)$ and then that the $g^{\mu\nu}\delta R_{\mu\nu}$ term contributes only total covariant derivative terms, $\nabla_\rho\nabla_\sigma(-\delta g^{\rho\sigma} + g^{\rho\sigma}g_{\alpha\beta}\delta g^{\alpha\beta})$, that can be dropped.

So we have, finally, Einstein's equations:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \qquad (4)$$

As a first check that things are good, note that energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$ is compatible with this equation, thanks to the Bianchi identity discussed last week, $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$.

Let's rewrite (4) another convenient way. Get $R - 2R = -R = 8\pi GT$, $T \equiv T^\mu_\mu$, so

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}). \qquad (5)$$

• Let's verify the Newtonian limit, where we'll take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and take $h_{\mu\nu}$ small. This will give a check on the $16\pi G$ in (1) and $8\pi G$ in (4). The 00 component of (5) is

$$R_{00} = 8\pi G(T_{00} - \tfrac{1}{2}Tg_{00}).$$

Let's check this for static matter, taking $T_{\mu\nu} \approx diag(\rho, 0, 0, 0)$, so

$$R_{00} \approx 4\pi G\rho.$$

Looking good for connecting with $\nabla^2\Phi = 4\pi\rho$. Indeed, use $R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)$, in the static Newtonian limit to get $R_{00} \approx R^i_{0i0} \approx -\frac{1}{2}\nabla^2 h_{00} = \nabla^2\Phi$, so the Newtonian limit checks.

• Aside on geodesic deviation. Recall

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu_{\nu\rho\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\delta x^\sigma.$$

Here we get

$$\frac{D^2}{D\tau^2}\delta x^i \approx R^i_{00j}\delta x^j = -R^i_{0j0}\delta x^j \approx -\partial^i\partial_j\Phi\delta x^j = -\delta(\partial^i\Phi),$$

fitting with the Newtonian picture that $\vec{a} = -\vec{\nabla}\Phi$.

• More generally, expanding around flat space by taking $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and linearizing in $h_{\mu\nu}$. Expand $R^\rho_{\sigma\mu\nu}$, and $R_{\sigma\nu}$, and R to linear order in small h. Discuss a bit now, more in the last lecture.

• Cosmological constant. There can be a constant term in S_{else} in (1),

$$S_{everything\ else} = S_{else} + \int d^4x\sqrt{-g}\left(\frac{-\Lambda}{16\pi G}\right). \tag{6}$$

The cc contributes to (3),

$$T^{cc}_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}. \tag{7}$$

So $\rho_{vac} = -p_{vac} = \Lambda/8\pi G$ (sometimes this is called the c.c.). Einstein's equations then become

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^{else}_{\mu\nu}. \tag{8}$$

This is how Einstein wrote it when he made his "greatest blunder" by putting it in to try to force his equations to give a static universe, rather than Hubble expansion.

• If there is mass density, e.g. inside a star, it contributes to the RHS of (5). Outside the star, assuming empty space, we have $T^{\mu\nu} = 0$, and thus (5) gives $R_{\mu\nu} = 0$. We can check that the Schwarzschild metric indeed has $R_{\mu\nu} = 0$, so it solves Einstein's equations outside the star.

If we take[1]

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2, \tag{9}$$

can compute $R_{\mu\nu}$ and see that it vanishes only if $\alpha = -\beta$ and $\partial_r(re^{2\alpha}) = 1$, which gives the Schwarzschild solution, $e^{2\alpha} = 1 - R_s/r$.

The Ricci tensor vanishes for Schwarzschild, but the Riemann tensor does not. Write out some example components, e.g. $R^r_{\phi r\phi} = re^{-2\beta}\sin^2\theta\partial_r\beta$, $R^t_{\theta t\theta} = -GM/r$, etc. The non-zero Riemann tensor will give e.g. the correct focusing of nearby geodesics,

$$\frac{D^2}{d\lambda^2}\delta x^\mu = R^\mu_{\nu\rho\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\delta x^\sigma.$$

Using the full metric, can explore this beyond the linearized limit discussed above.

Can show

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}.$$

It properly goes to zero far from the center of the star, as r^{-6}. We see that $r = 0$ is really a singularity. But, as we said before, $r = R_s$ is not a real singularity, only a fake coordinate one.

[1] a $e^{2\gamma(r)}$ factor in front of the $r^2 d\Omega^2$ term could be eliminated by a redefinition of $r \to e^\gamma r$