5/23/11 Lecture 17 outline

• Today we'll discuss Einsein's equations! Building blocks:

1. In the Newtonian limit, we should recover $\nabla^2\Phi = 4\pi G\rho$.
2. In the Newtonian limit, we should have $g_{00} \approx -(1+2\Phi)$.
3. General coordinate invariance.
4. Equivalence principle.

Einstein's equations are differential equations of motion for the metric $g_{\mu\nu}$, regarding it as a dynamical quantity. It is useful to get the equations of motion from an action:

$$S[g,X] = S_{grav}[g] + S_{\rm everythingelse}[g,X]. \tag{1}$$

Here X = all other matter or other fields in the problem, e.g. A^μ, etc. For short, we'll call the second term the matter contribution, but it includes everything. We could The equivalence principle states that

$$S_{matter}[g,X] = \int d^4x\sqrt{-|g|}\mathcal{L}_{matter}(\eta, X)|_{\eta\to g, \partial_\mu \to \nabla_\mu}, \tag{2}$$

i.e. the Lagrangian density is that of the theory without gravity, special relativitistic, with the simple replacement of $\eta_{\mu\nu} \to g_{\mu\nu}$. This fits with the fact that special relativity applies in local free-falling coordinates, $g_{\widehat{\mu\nu}} = \eta_{\mu\nu}$. The action (2) is general coordinate invariant thanks to the minimal replacement with properly transforming tensor quantities.

• Emphasize that (2) assumes and is equivalent to the equivalence principle. It'd be easy to violate this principle, e.g. coupling things like A_μ to the Riemann tensor. The equivalence principle says there are no such terms: curvature doesn't enter into (2). E.g. Maxwell's equations become $\nabla_\mu F^{\nu\mu} = J^\nu$, with no other corrections from gravity, via the metric, other than that hidden in ∇_μ.

• The gravity part of the action gives the terms involving derivatives of the metric. The basic, properly transforming tensor quantity with derivatives of the metric is the curvature. To make it generally coordinate invariant, the simplest possibility is

$$S_{grav} = S_{EH} = \frac{\alpha}{G}\int d^4x\sqrt{-g}R. \tag{3}$$

More complicated possibilities would include R^2 and higher terms, but they turn out to be not needed (though they might be there with small coefficients – and some theories like string theory can lead to some such terms – but no experimental sign of such other terms has shown up yet). Part of the reason for this is merely dimensional analysis. Recall from last time that we can take $[x] \sim L^0$ and $[g_{\mu\nu}] \sim L^2$ and then $[R] \sim L^{-2}$. Since $[S] \sim L^0$, and $[G] \sim L^2$, the parameter $[\alpha] \sim L^0$ is some number, that we'll determine. Higher curvature terms in (3) would be an expansion in powers of (GR), which is extremely small in normal circumstance, hence hard to observe.

- Now consider the E.L. EOM for (1), (2), (3):

$$\frac{\delta S_{tot}}{\delta g^{\mu\nu}} = \frac{\delta S_{EH}}{\delta g^{\mu\nu}} + \frac{\delta S_{matter}}{\delta g^{\mu\nu}} = 0. \tag{4}$$

The matter variation gives something nice because

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}. \tag{5}$$

(Recall energy and momentum of point particle from varying its endpoint. This is similar.)

It can be verified that (5) indeed agrees with what you'd get for the energy momentum tensor from the Noether procedure for spacetime translation symmetry (up to improvements that don't affect the conserved charges $P^\mu = \int d^3x T^{\mu 0}$). This is a nice byproduct of allowing the metric to vary: it gives a nicer way to define and compute the energy momentum tensor.

This is great, since we expect the energy momentum tensor to be the source term for $g'_{\mu\nu}s$ differential equation, so the Einstein equations will be

$$\tfrac{1}{2}\frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = T_{\mu\nu}. \tag{6}$$

Now we work out

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{\alpha}{G}\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\right). \tag{7}$$

Here there are three terms, from $\delta(\sqrt{-g}g^{\mu\nu}R_{\mu\nu})$. Show e.g. $\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}$, so $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. Also, get that $\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu} - (\lambda \leftrightarrow \nu)$ and then that the $\delta R_{\mu\nu}$ term contributes only total covariant derivative terms, that can be dropped.

- Finish next time.