• Today we'll discuss curvature. Last time we discussed covariant derivates ∇_μ, and mentioned that the Γ terms are analogous to the A^μ terms in the covariant derivatives of E& M. In this analogy, note that the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ arises as the commutator of E& M covariant derivatives

$$[\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = iqF_{\mu\nu}.$$

We'll see that the curvature is analogous to $F_{\mu\nu}$, and can be seen in the commutator of covariant derivatives, $[\nabla_\mu, \nabla_\nu] \neq 0$.

• Recall first that we can locally, at any point p, always find a coordinate transformation to "free-falling" coordinates, $x^{\widehat{\mu}}$, such that $g_{\widehat{\mu\nu}}|_p = \eta_{\widehat{\mu\nu}}$ and $\partial_{\widehat{\sigma}} g_{\widehat{\mu\nu}}|_p = 0$. We can thus locally make $\Gamma^{\widehat{\mu}}_{\widehat{\nu\sigma}}|_p$. At that point, the geodesic equation is $\frac{d^2 x^{\widehat{\mu}}}{d\tau^2}|_p = 0$.

But as we saw before in the example of the metric at the North pole of the earth, the 2nd order variations of the metric can't be set to zero. More generally, there is non-zero curvature at the point p if we can't set to zero the 2nd derivatives of the metric, or $\partial_{\widehat{\kappa}} \Gamma^{\widehat{\mu}}_{\widehat{\rho\sigma}}|_p \neq 0$.

• A related way to see curvature, as we mentioned last time, is that parallel transporting vectors gives a vector that depends on the path, e.g. you and your friend's vectors are no longer parallel when you parallel transport them from the equator to the North pole of the earth on different paths. This path dependence is an integrated version of the local statement that $[\nabla_\mu, \nabla_\nu]|_p \neq 0$ if there is local curvature at p

• Before getting into the details, let's do some counting of the expansion around the free-falling coordinates at p:

$$x^\mu = \frac{\partial x^\mu}{\partial x^{\widehat{\mu}}}|_p x^{\widehat{\mu}} + \tfrac{1}{2} \frac{\partial^2 x^\mu}{\partial x^{\widehat{\mu}} \partial x^{\widehat{\nu}}}|_p x^{\widehat{\mu}} x^{\widehat{\nu}} + \ldots,$$

and the metric has $g = \widehat{g}|_p + \widehat{\partial g}|_p \widehat{x} + \ldots$.

The metric $g_{\mu\nu}$ is a symmetric 4×4 matrix, so it has $4 \times 5/2 = 10$ independent components. We want to pick them to equal $\eta_{\widehat{\mu\nu}}$ at p, and that is done at this order by appropriate choice of the matrix $\frac{\partial x^\mu}{\partial x^{\widehat{\mu}}}$. The matrix $\frac{\partial x^\mu}{\partial x^{\widehat{\mu}}}$ is a 4×4 matrix, so 16 components, that we can use to set $g_{\widehat{\mu\nu}}|_p = \eta_{\widehat{\mu\nu}}$ (the extra, unused 6 components = the Lorentz group).

Now $\partial_\sigma g_{\mu\nu}$ has 40 terms, whereas $\frac{\partial^2 x^\mu}{\partial x^{\widehat{\mu}} \partial x^{\widehat{\nu}}}$ also has 40 terms, so this is just right to be able to set $\partial_{\widehat{\sigma}} g_{\widehat{\mu\nu}}|_p = 0$.

At the next order, $\partial_\lambda \partial_\sigma g_{\mu\nu}$ has 100 terms. But $\partial^3 x^\mu / \partial \widehat{x^\mu} \partial \widehat{x^\nu} \partial \widehat{x^\sigma}$ has $4(4{\cdot}5{\cdot}6)/3! = 80$ terms. So there are 20 independent components of $\partial_\lambda \partial_\sigma g_{\mu\nu}$ that can't be set to zero by any coordinate change. This will correspond to the 20 independent components of the Riemann curvature tensor.

- Let's now compute $[\nabla_\mu, \nabla_\nu]$ acting on a vector V^ρ, using

$$\nabla_\mu \nabla_\nu V^\rho = \partial_\mu \nabla_\nu V^\rho - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho, \qquad \nabla_\nu V^\rho = \partial_\nu V^\rho + \Gamma^\rho_{\nu\kappa} V^\kappa$$

and likewise for the term with μ and ν interchanged to get

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma,$$

where we used $\Gamma^\lambda_{[\mu\nu]} = 0$ and the Riemann curvature tensor is

$$R^\rho_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu).$$

By this construction, $R^\rho_{\sigma\mu\nu}$ transforms properly like a tensor. It's straightforward to directly verify that, under general coordinate transformations, the bad terms all cancel.

- Consider $R_{\rho\sigma\mu\nu} \equiv g_{\rho\lambda} R^\lambda_{\sigma\nu\nu}$. In local free-fall coordinates at p, get

$$R_{\widetilde{\rho\sigma\mu\nu}} = \tfrac{1}{2} \left[\left(\partial_{\widehat{\mu}} \partial_{\widehat{\sigma}} g_{\widehat{\rho\nu}} - (\rho \leftrightarrow \sigma) \right) - (\mu \leftrightarrow \nu) \right].$$

More generally, in any coordinate system, $R_{\rho\sigma\mu\nu} = R_{[\rho\sigma][\mu\nu]}$, antisymmetric in exchanging first two indices, or second two indices, $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ etc. Also, $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$. Also $R_{\rho[\sigma\mu\nu]} = 0$, $R_{[\rho\sigma\mu\nu]} = 0$. So the number of independent components in d spacetime dimensions is $\tfrac{1}{2}(\tfrac{1}{2}(d(d-1))(\tfrac{1}{2}d(d-1)+1) - (d!(d-4)!/4!)$, where the first factor accounts for $[\mu\nu]$, the second for symmetrizing in $([\mu\nu],[\rho\sigma])$, and the last for subtracting $R_{[\mu\nu\rho\sigma]} = 0$. The upshot is that Riemann curvature tensor in d spacetime dimensions has $d^2(d^2-1)/12$ independent components. In $d = 4$, this is 20 components, agreeing with our counting above.

- Can also show $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$, or in other words

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0,$$

this is a Bianchi identity, related to the Jacobi identity $[[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + \text{cyclic} = 0$. These can be easily shown in the free fall system.

- Define the Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = R_{\nu\mu}$, and the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$.

• Units: if we take $[x^\mu] \sim L^0$ and $[g_{\mu\nu}] \sim L^2$, then it follows immediately from the definitions in terms of the metric that $[\Gamma^\rho_{\mu\nu}] \sim [R^\rho_{\sigma\mu\nu}] \sim [R_{\mu\nu}] \sim L^0$, and $[R] \sim L^{-2}$.

• Defining $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, the Bianchi identity implies that $\nabla^\mu G_{\mu\nu} = 0$.

• Example, S^2: $ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$, taking $x^{A=1,2} = (\theta, \phi)$. Find $\Gamma^1_{22} = -\sin\theta\cos\theta$, $\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$, and others are zero. Then $R^1_{212} = \sin^2\theta$, $R_{1212} = a^2\sin^2\theta$, $R_{11} = 1$, $R_{12} = 0$, $R_{22} = \sin^2\theta$, and $R = 2/a^2$. "Constant curvature." Another example, R^3: $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ has $R^A_{BCD} = 0$, flat.

• If there is curvature, nearby geodesics deviate. Consider a geodesic $x^\mu(\lambda)$ and another $x^\mu + \delta x^\mu$. Recall that the geodesic equation is $\frac{D}{d\lambda}u^\mu \equiv u_\rho\nabla_\rho u^\mu = 0$. The difference, dropping terms higher than linear in δx^μ, satisfies

$$\frac{D^2}{d\lambda^2}\delta x^\mu = R^\mu_{\nu\rho\sigma}u^\nu u^\rho \delta x^\sigma,$$

where $u^\mu = dx^\mu/d\lambda$. The difference $\delta x^\mu(\lambda)$ thus doesn't stay constant if there is local curvature, e.g. the curvature due to dark matter can focus initially parallel light rays, gravitational lensing.

• Next, compute curvature of some of our favorite metrics, e.g. $ds^2_{Schwarzshild}$.