

## 5/16/11 Lecture 15 outline

- This week we'll return to some math topics, that we had deferred until now. As we've already discussed, GR is based on a local (a.k.a. gauge) symmetry principle: general coordinate transformations  $x^\mu \rightarrow x^{\mu'}(x)$ . Physical quantities and equations must transform properly under such transformations. Scalar quantities (e.g. mass, charge, proper time, action) are invariant,  $\phi(x') = \phi(x)$ . Quantities with indices, components of vectors or dual vectors, or more generally tensors, must transform with appropriate “ $\frac{\partial x}{\partial x}$ ” Jacobian matrix, e.g.  $A_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} A_\nu$  or  $T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} T^{\mu\nu}$ . Indices are raised or lowered with the metric.

We saw that, if  $\phi(x)$  is a scalar, then  $\partial_\mu \phi(x)$  transforms properly as a vector. We saw that if  $A_\nu(x)$  is a vector, then  $\partial_\mu A_\nu(x)$  does not quite transform as a tensor – there was a good term, but also a bad term, from the derivative acting on the Jacobian matrix. (On the other hand, we saw that  $\partial_\mu A_\nu - \partial_\nu A_\mu$  does transform properly, since the bad term cancels.)

This week we'll learn about covariant derivatives, which is a general way to cancel the bad term. It's analogous to E&M, where we replace  $\partial_\mu \rightarrow \partial_\mu + iqA_\mu$ , where the  $q$  is the charge of what it's acting on. As we'll discuss, the covariant derivative is  $\nabla_\mu = \partial_\mu + \dots$ , where the  $\dots$  account for the Lorentz indices of what it's acting on. For scalars,  $\nabla_\mu = \partial_\mu$ , that's like  $q = 0$  in the analogy. For vector, dual vector, or more generally tensors we need the  $\dots$  to cancel the bad terms, so  $\nabla_\mu$  acting on a tensor gives another tensor, both transforming properly. As we'll discuss, the  $\dots$  is something we've already met in our discussion of geodesics: the Christoffel connection.

- Write the geodesic equation

$$\frac{du^\mu}{d\lambda} + \Gamma_{\rho\sigma}^\mu u^\rho u^\sigma = 0, \quad u^\mu = \frac{dx^\mu}{d\lambda}.$$

Now  $u^\mu$  transforms properly as a 4-vector and  $\lambda$  (e.g. proper time for massive objects) transforms properly as a scalar. Recall that we derived the geodesic equation from  $\delta \int \sqrt{-ds^2} = 0$ , and  $\int \sqrt{-ds^2}$  transforms properly as a scalar. So, by construction, the geodesic equation must transform properly as a vector equation under general coordinate transformations  $x^\mu \rightarrow x^{\mu'}$ . This is a special case of a parallel transport equation.

- Parallel transport. Consider the change of a tensor quantity when we move it along some curve  $x^\mu(\lambda)$ . For example, consider the change of a vector quantity  $V^\mu(x(\lambda))$  when we

parallel transport it along the curve. If we change  $x^\mu \rightarrow x^{\mu'}$ , then we know  $V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$ . Now think of everything as depending on  $\tau$ , via  $x^\mu(\tau)$ . Then taking  $\frac{d}{d\tau}$  of both sides,

$$\frac{dV^{\mu'}}{d\tau} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{dA^\mu}{d\tau} + \frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\lambda} \frac{dx^\lambda}{d\tau} A^\nu.$$

Once again, there is a good term, and a bad term here, so the LHS doesn't transform properly. We can fix it as above. Let's discuss it more generally. As we transport the vector along the curve, There is some accompanying rotation,  $V^\mu(\lambda) = R^\mu_\nu(x(\lambda))V^\nu(0)$ , with the rotation depending on  $x^\mu$ , and we want it to be independent of the particular curve. So  $V(\lambda + d\lambda) = R(\lambda + d\lambda)R^{-1}(\lambda)V(\lambda) = [R(\lambda) + d\lambda \frac{dR}{d\lambda}]R^{-1}(\lambda)V(\lambda) = V(\lambda) + d\lambda \frac{dx^\mu}{d\lambda} \frac{\partial R}{\partial x^\mu} R^{-1}V(\lambda) = V(\lambda) + d\lambda \frac{DV}{d\lambda}$ . Adding in possible additional change to the vector, the change rate is  $\frac{DV^\mu}{d\lambda} = \frac{dV^\mu}{d\lambda} + \frac{\partial x^k}{d\lambda} \frac{\partial R^\mu_\sigma}{\partial x^k} R^{-1\sigma}_\lambda V^\lambda$ . In our case, we have

$$\frac{DV^\mu}{d\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} V^\sigma.$$

This transforms properly, with the Christoffel connection term canceling the bad term mentioned above. A vector is parallel transported if  $\frac{DV^\mu}{d\lambda} = 0$ . The geodesic equation can then be written simply as

$$\frac{Du^\mu}{d\lambda} = 0$$

i.e. the velocity is parallel transported.

The  $\Gamma$  term in the parallel transport accounts for the fact that moving vectors on curved spaces can lead to path dependent outcomes. For example, you and your friend start on the equator of the earth, each carrying a vector pointing due North. You carry your vector up to the North pole, keeping it always pointing North. Your friend first carries her vector 1/4 way around the earth, along the equator, and then brings it up to meet you at the North pole, again keeping it always pointing North. When you meet, you find that you're vectors are no longer parallel – even though you both parallel transported them. This effect is evidence of the curvature of the earth.

Parallel transporting is related to our desired covariant derivatives,

$$\frac{D}{d\lambda} = \frac{dx^\rho}{d\lambda} \nabla_\rho,$$

where this equation holds for acting on any kind of tensor whatsoever, with the appropriate definition of  $\nabla_\mu$ . When acting on a vector, we see from the above that  $\frac{dV^\mu}{d\lambda} = \frac{dx^\rho}{d\lambda} \nabla_\rho V^\mu$  with

$$\nabla_\rho V^\mu = \frac{\partial V^\mu}{\partial x^\rho} + \Gamma^\mu_{\rho\sigma} V^\sigma.$$

We can be sure that this will transform properly, because we can see its relation to the geodesic equation and we know that transforms properly. But let's check it explicitly:

$$\nabla_{\rho'} V^{\mu'} = \frac{\partial}{\partial x^{\rho'}} V^{\mu'} + \Gamma_{\rho'\sigma'}^{\mu'} V^{\sigma'},$$

where we now replace  $\frac{\partial}{\partial x^{\rho'}} = \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial}{\partial x^\rho}$  and  $V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$ , and likewise for  $V^{\sigma'}$ . Recalling that  $\Gamma_{\rho\sigma}^\mu = \frac{1}{2} g^{\mu\lambda} (\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma})$ , we can obtain

$$\Gamma_{\rho'\sigma'}^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \Gamma_{\rho\sigma}^\mu - \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} \frac{\partial^2 x^{\mu'}}{\partial x^\rho \partial x^\sigma}.$$

We call the first term good, because that's the proper transformation of the indices, and the second term bad. But that bad term, with its minus sign, is actually good: it's just what we wanted in  $\nabla_{\rho'}$ , because it precisely cancels the bad term that we mentioned before, from when  $\partial_{\rho'}$  acts on the  $\frac{\partial x^{\mu'}}{\partial x^\mu}$  in  $V^{\mu'}$ .

- How does  $\nabla_\rho$  act on  $V_\mu$ , with a lower index? Note that  $\phi = V_\mu V^\mu$  is a scalar, so  $\nabla_\rho \phi = V^\mu \nabla_\rho V_\mu + V_\mu \nabla_\rho V^\mu = \partial_\rho \phi$ . Alternatively, we can simply use  $V_\mu = g_{\mu\kappa} V^\kappa$ . Either way, we get

$$\nabla_\rho V_\mu = \frac{\partial V_\mu}{\partial x^\rho} - \Gamma_{\rho\mu}^\lambda V_\lambda.$$

The minus sign ensured that  $\nabla_{\rho'} V_{\mu'}$  transforms properly, with the bad terms canceling.

- The generalization to tensors is clear, we just treat each index as above e.g.

$$\nabla_\mu T^{\rho\sigma} = \partial_\mu T^{\rho\sigma} + \Gamma_{\mu\lambda}^\rho T^{\lambda\sigma} + \Gamma_{\mu\lambda}^\sigma T^{\rho\lambda}.$$

If  $T^{\rho\sigma}$  transforms properly, then so does this  $\nabla_\mu T^{\rho\sigma}$ .

- Can verify  $\nabla_\mu g_{\rho\sigma} = 0$ ; the metric is covariantly constant, thanks to the  $\Gamma$  terms. This is good, because we can then raise or lower indices on either side of the covariant derivatives, e.g.  $\nabla_\mu V_\lambda = \nabla_\mu (g_{\lambda\sigma} V^\sigma) = g_{\lambda\sigma} \nabla_\mu V^\sigma$ .

- Immediate point: conservation laws must transform properly. So charge conservation becomes

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma_{\mu\sigma}^\mu J^\sigma = 0.$$

Conservation of energy + momentum becomes

$$\nabla_\mu T^{\mu\sigma} = \partial_\mu T^{\mu\sigma} + \Gamma_{\mu\lambda}^\mu T^{\lambda\sigma} + \Gamma_{\mu\lambda}^\sigma T^{\mu\lambda} = 0.$$

These extra terms have a nice interpretation in terms of Stokes', Gauss' theorem. Let  $|g| \equiv -\det_{\mu\nu}(g_{\mu\nu})$  (the minus sign is just to cancel  $g_{00}$  being negative). The spacetime

integration measure  $d^4x$  doesn't transform properly as a scalar, because of the Jacobian determinant, but  $\sqrt{|g|}d^4x$  does. So all  $d^4x$ s of flat spacetime need to be replaced with  $\sqrt{|g|}d^4x$  in GR. Correspondingly,  $Q_{encl} = \int \sqrt{|g|}d^3x J^0$  is the conserved quantity.

This fits with the above conservation law, because you can verify that

$$\Gamma_{\mu\nu}^{\mu} = \frac{1}{\sqrt{|g|}}\partial_{\nu}\sqrt{|g|},$$

so find

$$\nabla_{\mu}J^{\mu} = \frac{1}{\sqrt{|g|}}\partial_{\mu}(\sqrt{|g|}J^{\mu}).$$