5/4/11 Lecture 12 outline

- Last time, precession of the perihelion + deflection of light, integrate

$$(\frac{dr}{d\phi})^2 + 2\frac{r^4}{\ell^2}V_{eff}(r) = \frac{r^4}{\ell^2}e^2,$$

to get

$$\Delta\phi = \int dr\frac{d\phi}{dr} = \int dr\frac{\ell}{r^2}\frac{1}{\sqrt{e^2 - 2V_{eff}(r)}}.$$

$$V_{eff}(r) = \tfrac{1}{2}\epsilon - \frac{\epsilon GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\gamma\ell^2}{r^3}.$$

For an orbit, between the inner and outer turning points (the zeros of $e^2 = 2V(r)$), get

$$\Delta\phi = 2\int_{r_1}^{r_2} dr\frac{\ell}{r^2}\frac{1}{\sqrt{e^2 - 2V(r)}}.$$

In the Newtonian case, $\gamma = 0$, can do the integral, get $\Delta\phi = 2\pi$, so the orbits come back to themselves. For $\gamma = 1$, get $\Delta\phi > 2\pi$, so they overclose, and the perihelion precesses.

Let $x = \ell^2/GMr$, in the equation we're integrating above, and take $\frac{d}{d\varphi}$ of that equation to obtain

$$\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2\gamma x^2}{\ell^2}.$$

If $GM/\ell \ll 1$, we can treat the last term as a perturbation, $x = x_0 + x_1$, with $x_0 = 1+\xi\cos\phi$, where $\xi^2 = 1-b^2/a^2$ is the eccentricity of the ellipse, usually called e. Then find $x \approx 1+\xi\cos(1-\alpha)\phi$, and $\Delta\phi \approx 2\pi(1+\alpha)$, where $2\pi\alpha \approx 6\pi G^2M^2/\ell^2 \approx 6\pi GM/c^2a(1-\xi^2)$. For Mercury, use $GM_{sun}/c^2 = 1/48 \times 10^5 cm$, $a = 5.79 \times 10^{12} cm$, $\xi = 0.2056$, to get $2\pi\alpha \approx 5.01 \times 10^{-7}$ radians per orbit, which had been observed before (!) the GR calculation.

Deflection of light (radially, comes in and bounces off the $V_{eff}(r)$ barrier):

$$\Delta\phi = 2\ell\int_{r_1}^{\infty}\frac{dr}{r^2}\frac{1}{\sqrt{e^2 - 2V(r)}}, \qquad V(r) = \frac{\ell^2}{2r^2} - \frac{\gamma GM\ell^2}{r^3}$$

Using $\ell/e = b$ and defining $w = b/r$ get

$$\Delta\phi = 2\int_0^{w_1} dw(1 - w^2(1 - \frac{2GMw\gamma}{b}))^{-1/2}.$$

For $\gamma = 0$, get $\Delta\phi = \pi$, no deflection in Newtonian case. For $\gamma = 1$, get $\Delta\phi > \pi$, corresponding to focusing. Approximating $GM/b \ll 1$, get $\Delta\phi \approx \pi + \frac{4GM}{b}$. Deflection becomes infinite for $GM/b \to 1/\sqrt{27}$, which is where we already saw last time is the b_{crit} needed to overcome the barrier in $V_{eff}(r)$ (at $r = 3GM$). Also time delay of light in GR.

- Next topic, parameterized Post Newtonian (PPN) parameters. Let $ds^2 = -A(r)(cdt)^2 + B(r)dr^2 + r^2 d\Omega^2$, with

$$A(r) = 1 - \frac{2GM}{c^2 r} + 2(\beta - \gamma)\left(\frac{2GM}{c^2 r}\right)^2 + \ldots,$$

$$B(r) = 1 + 2\gamma\left(\frac{2GM}{c^2 r}\right) + \ldots,$$

where GR has $\gamma = 1$ and $\beta = 1$. The parameters γ and β affect the deflection of light, precession of the perihelion, and the time delay of light, e.g. $\delta\phi_{delf} \approx \frac{1}{2}(1+\gamma)(4GM/c^2 b)$ (so the Newtonian limit isn't $\gamma = 0$, this γ differs from the one used above). They also affect the GPS system. They are thus experimentally measurable quantities, e.g. γ can be measured by lensing radio sources behind the sun. Observation shows $\beta, \gamma = 1.000 \pm \mathcal{O}(10^{-4,-5})$.

- Consider the Schwarzschild metric when $r_{object} \leq R_S = 2GM$: the Schwarzschild black hole. Consider null, radial geodesics, see they have $dt/dr = \pm(1 - \frac{2GM}{r})$, so the slope of the light cones in the (r, t) plane close up at $r = R_S = 2GM$. A light ray just outside that radius seems to never get there, but this is an illusion of the coordinate system.