5/2/11 Lecture 11 outline

• Finish up geodesics in the Schwarzschild metric:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -(1-2GM/rc^2)dt^2 + (1-2GM/rc^2)^{-1}dr^2 + r^2 d\Omega^2$$

The conserved quantities for a particle with 4-velocity u^μ are

$$e \equiv -\xi \cdot u = (1-\frac{2M}{r})\frac{dt}{d\tau}, \qquad \ell \equiv \eta \cdot u = r^2 \sin^2\theta \frac{d\phi}{d\tau}$$

which are the energy and angular momentum per unit mass, respectively, as seen by an observer at $r=\infty$, and $u \cdot u = -\epsilon$, where $\epsilon \equiv 1$ for massive objects and $\epsilon \equiv 0$ for massless ones. We found that the radial motion is given by $\frac{1}{2}e^2 = \frac{1}{2}(\frac{dr}{d\lambda})^2 + V_{eff}(r)$,

$$V_{eff}(r) = \tfrac{1}{2}\epsilon - \frac{\epsilon GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\gamma\ell^2}{r^3}.$$

For a massive object we can multiply the above by m and use $L = \ell m$ to make the first two terms look familiar. The first term is the Newtonian potential, there only for massive objects. The second term is the angular momentum barrier, there for both massive and massless objects. The third term has $\gamma_{GR} = 1$, and $\gamma_{Newtonian} = 0$;since its $\sim 1/r^3$ its negligible away from the origin but it dominates for sufficiently small r. It replaces the infinite centrifugal barrier of Newtonian mechanics with a barrier of finite height.

• Draw pictures for timelike (massive) and null (massless) cases, compare / contrast with Newtonian case. For a massive object, the shape of $V_{eff}(r)$ depends on the size of ℓ. For a massless object, ℓ affects only the overall scale size of $V_{eff}(r)$, not its shape.

• Look for circular orbits, $dV/dr = 0$: $\epsilon MGr_c^2 - \ell^2 r_c + 3GML^2\gamma = 0$. For massless case, $\epsilon = 0$, no solution for $\gamma = 0$, but for $\gamma = 1$ get $r_c = 3GM$. This is a local maximum, unstable to perturbations. For the massive case, $\epsilon = 1$, get

$$r_c = \frac{\ell^2 \pm \sqrt{\ell^4 - 12GM^2\ell^2}}{2GM}.$$

For $\ell^2 > 12GM^2$, the inner one is unstable and the outer one is stable. For $\ell \gg 1$ get $r_c \approx \ell^2/GM$, which is the stable Newtonian result, and $r_c = 3GM$, which is unstable.

For $\ell^2 = 12GM^2$, there is only 1 orbit, at $r_c = 6GM$. This is the smallest possible stable orbit. For $\ell^2 < 12GM^2$, there are no extrema of V_{eff}, the potential just slides down, down, down to the singularity at $r = 0$, goodbye.

• Consider the null case. The minimum e needed to climb the barrier is given by $\frac{1}{2}e^2 = V_{eff}(r = 3GM) = \ell^2/2(27)(GM)^2$, or $\ell^2/e^2 = 27(GM)^2$. At infinity, we have $\ell = be$, where b is the impact parameter. To see that note that, at infinity, $\ell/e = r^2 \frac{d\phi}{d\lambda}/(1 - \frac{2GM}{r})\frac{dt}{d\lambda} \to r^2 d\phi/dt$ and so $\phi \approx b/r$, with $dr/dt \approx -1$. So we see that light with impact parameter less than $b_c = 3^{3/2}GM$ is captured. The capture cross section is $\sigma_c = 27\pi(GM)^2$.

• Study precession of the perihelion + deflection of light, multiplying the radial equation by $(d\lambda/d\phi)^2 = r^4/\ell^2$ to convert $(dr/d\lambda)^2$ there into $(dr/d\phi)^2$:

$$(\frac{dr}{d\phi})^2 + 2\frac{r^4}{\ell^2}V_{eff}(r) = \frac{r^4}{\ell^2}e^2.$$

So

$$\frac{d\phi}{dr} = \pm\frac{\ell}{r^2}\frac{1}{\sqrt{e^2 - 2V(r)}}.$$

For an orbit, between the inner and outer turning points (the zeros of $e^2 = 2V(r)$), get

$$\Delta\phi = 2\int_{r_1}^{r_2} dr \frac{\ell}{r^2}\frac{1}{\sqrt{e^2 - 2V(r)}}.$$

In the Newtonian case, $\gamma = 0$, can do the integral, get $\Delta\phi = 2\pi$, so the orbits come back to themselves. For $\gamma = 1$, get $\Delta\phi > 2\pi$, so they overclose, and the perihelion precesses.