4/27/11 Lecture 10 outline

• Continue, we're going to solve for geodesics in the Schwarzschild metric:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -(1-2GM/rc^2)dt^2 + (1-2GM/rc^2)^{-1}dr^2 + r^2 d\Omega^2$$

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0,$$
$$\Gamma^\nu_{\mu\sigma} = \tfrac{1}{2}g^{\rho\nu}(\partial_\mu g_{\rho\sigma} + \partial_\sigma g_{\rho\mu} - \partial_\rho g_{\mu\sigma})$$

These equations describe the motion of an object – regardless of its mass (equivalence principle, but assuming it's not so massive as to change the metric), it can be an apple or a photon – in the background spacetime geometry created by some mass M.

The equations are complicated and its best to make use of the symmetries, just like in Newtonian mechanics where we utilize rotational and time translational symmetry.

• The Schwarzschild metric is time independent and spherically symmetric. These symmetries imply conservation laws for particles moving on geodesics. Using (t, r, θ, ϕ) as our coordinates, we write the Killing vectors as $\partial_t \to \xi^\mu = (1, 0, 0, 0)$ and $\partial_\phi \to \eta^\mu = (0, 0, 0, 1)$ for time translations and rotations in ϕ, respectively. The conserved quantities for a particle with 4-velocity u^μ are

$$e \equiv -\xi \cdot u = (1 - \frac{2M}{r})\frac{dt}{d\tau}, \qquad \ell \equiv \eta \cdot u = r^2 \sin^2\theta \frac{d\phi}{d\tau}.$$

Here e is like the energy per unit mass and ℓ is like the angular momentum per unit mass. Note however that e isn't the "KE" seen by a stationary observer, at fixed spatial position. Such an observer measures the test object to have $\widehat{E}_{obj;obs} = -g_{\mu\nu}u^\mu_{object}u^\nu_{observer}$. (Here we use the hat to mean energy per unit mass.) If the observer is stationary only $u^0_{obs} \neq 0$, and is given by $-g_{00}(u^0_{obs})^2 = -1$, so $u^\mu_{obs} = (-g_{00})^{-1/2}\xi$, and thus $\widehat{E}_{obj;obs} = e_{obj}(-g_{00})^{-1/2}$. Now e_{obj} is a constant, but g_{00} depends on r, so a stationary observer at fixed r measures $E_{obj,obs}$ depending on r. At $r \to \infty$, $g_{00} \to -1$, so $e_{obj} = E_{obj,\infty}$, the energy observed by an observer at infinity. To summarize,

$$E_{obj,obs}(r) = E_{obj,\infty}(1 - \frac{2GM}{r})^{-1/2}, \qquad \widehat{E}_{obj,\infty} \equiv e$$

• This applies whether the observer is measuring the energy of an apple or a photon. For photons, it implies the gravitational redshift

$$\omega_\gamma(r) = \omega_\gamma(\infty)(1 - \frac{2GM}{r})^{-1/2}.$$

Therefore, a stationary observer at radius r_1 will measure the photon as having frequency $\omega_1 = \omega(r_1)$ and another at radius r_2 will measure it as having frequency $\omega_2 = \omega(r_2)$, with

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - 2GM/r_1}{1 - 2GM/r_2}\right)^{1/2}.$$

E.g. for $r \gg 2GM$ this gives $\omega_2/\omega_1 \approx 1 + \Phi_1 - \Phi_2$, which is what we saw before from the rocket picture. This formula makes sense for $r > 2GM$. A photon starting at $r = 2GM$ would be redshifted to zero frequency by the time it gets to infinity – in other words, it can't make it out.

We can also use this to determine the escape velocity needed for a massive object, starting at fixed r, to make it to infinity. (For massless objects, there's no notion of escape velocity – it can always make it to infinity from any $r > 2GM$. For $r \leq 2GM$, the light doesn't escape, as we saw from the redshift formula. To make it to infinity, need $e = 1$, so the observer at fixed r needs to see the object as having

$$\widehat{E}_{obj,obs,esc} = (1 - \frac{2GM}{r})^{-1/2} \equiv (1 - V_{esc}^2/c^2)^{-1/2},$$

so $V_{esc} = \sqrt{2M/R}$, coincidentally the same as in Newtonian mechanics. For $r \to 2GM$, get $V_{esc} \to c$.

- Back to the geodesic equations. Use constants e, ℓ, and also $u \cdot u = -\epsilon$, where $\epsilon \equiv 1$ for massive objects and $\epsilon \equiv 0$ for massless ones. Conservation of angular momentum implies that orbits lie in a plane. E.g. if the particle is moving with $d\phi/d\tau$ at an instant, then $\ell = 0$ for all time. Instead take $\theta = \pi/2$ and $u^\theta = 0$, and it remains so for all time.

$$-\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1} (u^r)^2 + \ell^2/r^2 = -\epsilon.$$

Define $\mathcal{E} \equiv \frac{1}{2}(e^2 - \epsilon)$ and then the equation can be written in a familiar form, $\mathcal{E} = \frac{1}{2}(\frac{dr}{d\lambda})^2 + V_{eff}(r)$, with

$$V_{eff}(r) = -\frac{\epsilon GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}.$$

For a massive object we can multiply this by m and use $L = \ell m$ to make the first two terms look familiar. The first term is the Newtonian potential, there only for massive objects. The second term is the angular momentum barrier, there for both massive and massless objects. The third term is a new contribution when $\ell \neq 0$, and since its $\sim 1/r^3$

its negligible away from the origin but it dominates for sufficiently small r. It replaces the infinite centrifugal barrier of Newtonian mechanics with a barrier of finite height.

• Next time: Draw pictures for massive and massless cases, compare / contrast with Newtonian case.

• Look for circular orbits, $dV/dr = 0$: $\epsilon MGr_c^2 - \ell^2 r_c + 3GML^2\gamma = 0$. Here $\gamma = 1$ for GR vs $\gamma = 0$ for Newtonian. For massless case, $\epsilon = 0$, no solution for $\gamma = 0$, but for $\gamma = 1$ get $r_c = 3GM$. This is a local maximum, unstable to perturbations. For the massive case, $\epsilon = 1$, get

$$r_c = \frac{\ell^2 \pm \sqrt{\ell^4 - 12GM^2\ell^2}}{2GM},$$

where the inner one is unstable and the outer one is stable.

For $\ell \gg 1$ get $r_c \approx \ell^2/GM$, which is the stable Newtonian result, and $r_c = 3GM$, which is unstable.

• Study precession of the perihelion + deflection of light, multiplying the radial equation by $(d\lambda/d\phi)^2 = r^4/\ell^2$ to convert $(dr/d\lambda)^2$ there into $(dr/d\phi)^2$:

$$(\frac{dr}{d\phi})^2 - 2\epsilon gMr^3/\ell^2 + r^2 - 2GMr\gamma = 2\mathcal{E}r^4/\ell^2.$$