Physics 225, Homework 5, Due Wednesday June 1.

1. Consider the 2-sphere with coordinates $x^A = (\theta, \phi)$ and metric

$$dS^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Take a vector with components $V^A = (1, 0)$ (i.e. $V = \frac{d}{d\theta}$) and parallel-transport it once around a circle of constant latitude, i.e. a constant value of $\theta$. What are the components of the resulting vector, as a function of $\theta$?

2. Consider the 4d spacetime with metric

$$ds^2 = -dt^2 + t^{4/3}(dx^2 + dy^2 + dz^2),$$

(which is a "matter-dominated Freedman Robertson Walker metric"). A certain vector, at coordinate time $t$, has components $V^\nu = (5t^2, 7t^3, 0, 0)$ (only the $t$ and $x$ components are non-zero). Write out the 16 components of the tensor $\nabla_\mu V^\nu$ at coordinate time $t$.

3. Calculate the Riemann curvature for the metric

$$ds^2 = -(1 + Cx)^2 dt^2 + dx^2 + dy^2 + dz^2.$$

Here $C$ is a constant (this is roughly like a constant gravitational force). Is this space curved or flat? Do nearby geodesics deviate from each other?

4. Consider the metric $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, where the latin indices $i, j$ run from $1 \ldots 3$ (corresponding to $x$, $y$, and $z$). Compute the Riemann tensor components $R_{0i0j}$ and $R_{ijkl}$ (0, of course, refers to $t$). Compute the Ricci tensor components $R_{00}$, $R_{i0}$ and $R_{ij}$. Finally, compute the Ricci scalar $R$. The answers for these tensors are quoted in many references, so I want to see some of your work.

5. Consider

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\right)$$

ordinary Maxwell theory, coupled to source $J_\mu$, and coupled to the metric according to the equivalence principle.

(a) Compute $T_{\mu\nu}$ by functional differentiation with respect to the metric.

(b) Now add the equivalence violating term

$$\mathcal{L}' = \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}.$$

How are Maxwell's equations modified in the presence of this term. Is the current still conserved?

(c) **Optional** exercise: work out how Einstein's equations are modified by the above $\beta$ term.