Physics 225, Homework 2, Due Monday April 11.

1. As discussed in lecture, for a collection of particles (labeled by n) we have

$$T^{\mu\nu}(\vec{x},t) = \sum_n \frac{p_n^\mu p_n^\nu}{E_n}\delta^3(\vec{x}-\vec{x}_n(t)).$$

Suppose that the particles have zero rest mass (e.g. photons) and imagine that one smooths over the delta function to write $T^{\mu\nu}$ as for a perfect fluid. (The idea is to replace the sum of delta functions with a continuous distribution which is commonplace in physics, e.g. in going from the $\rho(\vec{x})$ of many point particles to a smooth charge distribution.) Without worrying about the details of how one actually goes about smoothing out the sum of delta functions, argue that the resulting "fluid" has $p = \frac{1}{3}\rho$.

2. The Maxwell field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B_3 & -B_2 \\ -E^2 & -B_3 & 0 & B_1 \\ -E^3 & B^2 & -B_1 & 0 \end{pmatrix}.$$

Write out the tensor $\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Here $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is Levi-Civita tensor density; the tilde is to distinguish it from a related quantity that's an actual tensor, but we're not going to make that distinction now (see Carroll eqn. 1.68), with $\tilde{\epsilon}^{0123} = 1$ and all even permutations of 0123 also giving $+1$, and all odd permutations giving -1. Write out $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Show that the $\nabla\cdot\vec{B}$ and $\nabla\times\vec{E}$ Maxwell equations can be written as $\partial_\mu \widetilde{F}^{\mu\nu} = 0$.

3. Suppose that the stress energy tensor in a local intertial frame is $T^{\mu\nu} = diag(A,B,C,D)$. What condition on A, B, C, D must be satisfied so that any observer (moving with arbitrary velocity, though of course less than c, relative to the above frame will observe a **positive** energy density. Hint: There is a general expression, in terms of $T^{\mu\nu}$ and u^μ, for the energy density measured by an observer with 4-velocity u^μ; use it. Consider the special case $T^{\mu\nu}_{cosmo\ const} = -A\eta^{\mu\nu}$, where $B = C = D = -A$, with $A > 0$. Is there any frame where the energy density is negative?

4. The plane $z = 0$ has surface charge density σ. The plane $z = h$ has the opposite surface charge density. There are no current densities in this lab reference frame.

a) Find the surface charge density σ' and surface current $\vec{K}'$ on those surfaces as seen by an observer in a rocket, moving along the $+x$ axis with velocity v.

b) Find the electric and magnetic fields $\vec{E}$ and $\vec{B}$ everywhere in space (using Maxwell eqns in the units written as in class, without the μ_0 and ϵ_0 of MKS, and with $c = 1$), according to an observer in the lab frame. Using your answer to part (a), compute $\vec{E}'$ and $\vec{B}'$ as measured by an observer in the rocket frame. Now verify that the relation you found between the fields in the two frames of reference agree with the Lorentz transformation formula $F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}F^{\rho\sigma}$.

5. (Taken from Hartle 6.13) Three observers are standing near each other at the surface of the Earth. Each holds an accurate atomic clock. At time $t = 0$ the clocks are all synchronized. At $t = 0$, the first observer throws his clock straight up, so that it returns at time T as measured by the clock of the second observer, who holds her clock in her hand for the entire time interval. The third observer carries his clock up to the same maximum height the thrown clock reaches, and back down, moving with constant speed on each leg of the trip and returning in time T (as measured by the clock of the second observer). Calculate the total elapsed time measured on each clock, assuming that the maximum height is much smaller than the radius of the Earth. Include gravitational effects but calculate only to order $1/c^2$, using nonrelativistic trajectories. Which clock registers the longest time? Why is this?