Physics 220, Lecture 9

⋆ Reference: Georgi chapters 2 and 3.

• Last time: Representations $D_r(g)$. Generators in representation, $T^A_r = -i\partial_{\alpha_A} D(\alpha)|_{\alpha=0}$. Write $D_r(g(\alpha)) = e^{i\alpha_A T^A_r}$. Compact vs non-compact groups, examples. Now we'll call $|G|$ the number of generators.

• Example of $SO(2)$. $U(\phi) = e^{i\phi T}$. Defining representation (reducible). Irreps all 1d (abelian), with $T|m\rangle = m|m\rangle$, so $U_m(\phi) = e^{im\phi}$. Defining rep is $D_1 \oplus D_{-1}$. Character orthogonality and completeness:

$$\int_0^{2\pi} \frac{d\phi}{2\pi} U_n^\dagger(\phi) U_m(\phi) = \delta_{nm}, \qquad \sum_m U_m(\phi) U_m^\dagger(\phi') = \delta(\phi - \phi').$$

$U_m(\phi) = \langle\phi|m\rangle$; in $\langle\phi|$ basis $T \to i\frac{d}{d\phi}$. On to general, non-Abelian groups!

• Group multiplication law near $\alpha = 0$: $e^{i\alpha_A T^A} e^{i\beta_B T^B} = e^{i(\alpha+\beta)_A T^A - \frac{1}{2}[\alpha T, \beta T] + \ldots)}$, i.e. the "Lie algebra": $[\alpha T, \beta T] = i\gamma_C T^C$. This requires

$$[T^A, T^B] = i f_{ABC} T^C,$$

and then $\gamma_C = \alpha_A \beta_B f_{ABC}$. The algebra accounts for what happens in a tiny region near the origin. It turns out that this is enough to get the group multiplication law everywhere, by exponentiation. The structure constants f_{ABC} essentially specify the group. Jacobi identity. If there is a unitary rep, the T^A are hermitian and then it follows that the f_{ABC} **are real**, since $[T^A, T^B]^\dagger = [T^B, T^A]$. Can normalize by $Tr(T_a T_b) = k_a \delta_{ab}$, where $k_a > 0$ for compact Lie algebras, and can take all $k_a = \lambda$. Then $f_{abc} = -i\lambda^{-1}\mathrm{Tr}([T^a, T^b]T^c)$ is completely antisymmetric in the 3 indices. Simplest case: $f_{ABC} = \epsilon_{ABC}$, with $A = 1 \ldots 3$.