Physics 220, Lecture 6

• Aside from last time: the quaternion group can be described as generated by elements a and b, with $a^4 = e$, $b^4 = a^2$, $ba = ab^3$. On the other hand, D_4 is generated by elements a and b with $a^4 = b^2 = (ba)^2 = e$. These groups differ, but have the same character table.

• Consider a physical situation with some symmetry group G, and where we want to solve an eigenvalue equation like $L\psi = \lambda\psi$. For example, this could the equation for some normal modes of vibration. Or it could be the Schrodinger equation. The eigenvectors ψ must form a (generally reducible) representation of G, $\psi_{a,K_a,i}$, where a labels some irrep, $K_a = 1 \dots m_a$ labels copies of that irrep, and i is the basis within the irrep, $i = 1 \dots n_a$. If G is non-Abelian, it has some irreps of dimension $n_a > 1$, and that shows up as a degeneracy among the corresponding eigenvalues: $\lambda = \lambda_{a,K_a}$ are independent of $i = 1 \dots n_a$.

• Physics example: three blocks connected by springs (Georgi). Normal modes, solve for $\lambda = m\omega^2/k$. Show $D = D_0 + D_1 + 2D_2$. Corresponding normal modes λ_0, λ_1, $\lambda_{2,1}$, $\lambda_{2,2}$. Show $\lambda_0 = 3$ (breathing mode); $\lambda_1 = 0$ (rotation mode); $\lambda_{2,1} = 0$ (translation modes); $\lambda_{2,2} = \frac{3}{2}$.

• Physics example: drum head perturbed by 4 masses forming a square. Before perturbation, wave equation has degenerate solutions: $\psi_1 = J_m(kr)\cos m\theta e^{-i\omega t} \sim \cos m\theta$, and similarly for $\psi_2 \sim \sin m\theta$. These two are degenerate, and that is ensured bythe fact that they form a $n_a = 2$ dimensional representation of the symmetry group $O(2)$. The perturbing masses break the symmetry group $O(2)$ to D_4. Decompose the 2d $O(2)$ rep into irreps of D_4 using characters to see if the perturbation splits the degeneracy. Character: $\chi(e) = 2$, $\chi(C) = 2\cos(m\pi/2)$, $\chi(C^2) = 2\cos(m\pi)$, $\chi(\sigma) = \chi(\sigma C^2) = 0$. So irrep for m odd, and reducible for m even. So for m even the degeneracy can be lifted, since the two irreps aren't related by symmetry. Whereas for m odd the degeneracy isn't lifted, since the states are related by the D_4 symmetry.

• Symmetry of $2n+1$-gon, D_{2n+1}. The classes are e; all reflections σ; rotations R_j by $\pm 2\pi j/(2n+1)$ for $j = 1 \dots n$, so $n+2$ classes. Therefore also $n+2$ irreps, of dimensions D_0, D_1, $D_{2.m}$, $m = 1 \dots n$, with $|G| = 2(2n+1) = 1^2 + 1^2 + n(2)^2$. Character table, $\chi_1(\sigma) = -1$, $\chi_1(R_j) = 1$, $\chi_{2,m}(\sigma) = 0$, $\chi_{2,m}(R_j) = 2\cos(2\pi mj/(2n+1))$. Verify character orthogonality relations. Verify e.g. $D_{2,1} \otimes D_{2,1} = D_0 + D_1 + D_{2,2}$.

• S_4 character table. Mention Young tableau for conjugacy classes (as before), and now also irreps.