Physics 220, Lecture 5

• Last time:

$$A^{ab}_{jl} \equiv \sum_{g\in G} D_a(g^{-1})|a,j\rangle\langle b,l|D_b(g) = \frac{|G|}{|r_a|}\delta_{ab}\delta_{jl}I.$$

Implies that $\sqrt{\frac{|r_a|}{|G|}}[D(g)]_{jk}$ are orthonormal functions of the group elements g. Also complete set.

• Characters. $\chi_D(g) = \mathrm{Tr}D(g)$, satisfy an orthogonality relatiosn:

$$\frac{1}{|G|}\sum_{g\in G}\chi_{D_a}(g)^*\chi_{D_b}(g) = \delta_{ab},$$

and there is another orthogonality condition, involving the sum over irreps:

$$\sum_a \chi_{D_a}(g_\alpha)^*\chi_{D_a}(g_\beta) = \frac{|G|}{k_\alpha}\delta_{\alpha\beta},$$

where α and β label the conjugacy classes, and k_α is the number of elements in that class.

The characters provide a complete set of class functions: any function of only conjugacy classes can be expanded in terms of the characters. As we already said,

$$F(g_1) = \sum_{a,j,k} c^a_{jk}[D_a(g_1)]_{jk}$$

Now on both sides replace g_1 with $g^{-1}g_1g$ and average over all g. Use the orthogonality relations to do the g sum and get finally

$$F(g_1) = \sum_{a,j}\frac{1}{|r_a|}c^a_{jj}\chi_a(g_1).$$

• Example: Z_3 and S_3 character tables again. (Get S_3 table 3 ways: using explicit expressions for irreps; using character orthogonality; using $g^p = e$ implies character is a sum of $|r_a|$ eigenvalues each satisfying $\lambda_i^p = 1$.)

• Any rep D can be written as a sum of irreps, $D = \sum_a m^D_a D_a$, and can find m^D_a by

$$m^D_a = \frac{1}{|G|}\sum_{g\in G}\chi_{D_a}(g)^*\chi_D(g).$$

Note that $\sum_{g\in G}\chi^*_D(g)\chi_D(g) = |G|\sum_a m_a^2$, so the rep is an irrep iff $\sum_{g\in G}\chi^*_D(g)\chi_D(g) = |G|$. Gives a quick test to see if a rep is an irrep. For example, the regular rep has $\sum_{g\in G}\chi^*_D(g)\chi_D(g) = |G|^2$, so not an irrep.

Example: show that the regular rep has $m_a^R = |r_a|$. Fits with $|G| = \sum_a |r_a|^2$.

Construct the projection operators for non-irreps:

$$P_a = \frac{|r_a|}{|G|} \sum_{g \in G} \chi_{D_a}(g)^* D(g).$$

This projects on D_a. To see that, replace $D(g)$ with $[D_b(g)]_{jk}$ and get $\delta_{ab}\delta_{jk}$.

Example: 3d representation of S_3. Construct projection operators to show how $D_3 = D_0 + D_2$.

• Decomposing tensor products of irreps into sums of irreps , e.g. for S_3 ($D_2 \times D_2 = D_0 + D_1 + D_2$ etc.).