Physics 220, Lecture 3

• Definition of a representation: $D(g)$ as linear operators on vector spaces, wtih $D(g_1)D(g_2) = D(g_1 g_2)$. Choosing a basis for the vector space, $|i\rangle$, then the idea is that g maps $|i\rangle \to \sum_j |j\rangle\langle j|D(g)|i\rangle$, so group multiplication becomes matrix multiplication.

Examples: The trivial representation. Three 1d representations of Z_3, and its 3d representation. Examples of 1d and 2d S_3 representations, $D_0 = 1$, D_1 from ϵ^{ijk}, and D_2 obtained from subgroup of $O(2)$ ($D_2(a_1) = R(2\pi/3)$ and $D_2(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$).

• The "regular" representation for any group: form $|g_i\rangle$ then $D(g_1)|g_2\rangle = |g_1 g_2\rangle$.

• A rep is reducible if all $g \in G$ map some subspace of the vector space onto itself. In other words, reducible if there is a projection operator P such that $PD(g)P = D(g)P$ for all $g \in G$.

• The regular rep is always reducible, since $\frac{1}{|G|}\sum_{g\in G}|g\rangle$ gives an invariant subspace. Example: the 3d rep of Z_3 has diagonal $D'(g) = S^{-1}D(g)S$, give S. Shows Reg=$D_1 + D_2 + D_3$, a direct sum of 3 1d representations.

Irreducible if not reducible; these are the basic reps (we'll later see a connection with conjugacy classes). The 1d reps are irreps, and the 2d rep of S_3 is also an irrep. Give all irreps of Z_N, all 1d; non-abelian groups have some $d > 1$ irreps.

Completely reducible if equivalent (under $D(g) \to D'(g) = S^{-1}D(g)S$ basis change) to a direct sum of irreducible representations. An example of a reducible but not completely reducible rep is $D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ which represents e.g. addition of integers (or translations). It is reducible because $D(x)P = P$ where $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, but not completely reducible since $D(x)(1-P) \neq (1-P)$. This never happens for finite groups.

• Finite groups: all reps equivalent to unitary ones. Prove this by construction: given any rep $D(g)$, show that $D'(g) = XD(g)X^{-1}$ is unitary, where $X^2 \equiv S \equiv \sum_{g\in G} D(g)^\dagger D(g)$. Note that S is hermitian, $S^\dagger = S$, and $S = U^{-1}DU$, where D is a diagonal matrix of eigenvalues, which are positive and can't be zero (a zero eigenvalue would imply a $|v\rangle$ with $\langle v|S|v\rangle = \sum_{g\in G} ||D(g)|v\rangle|| = 0$, which is impossible since $D(e) = 1$); this justifies the existence of X and X^{-1}. Then show $D'(g)^\dagger D'(g) = 1$.

• For finite groups, any reducible rep is fully reducible. For a reducible unitary representation, $PD(g)P = D(g)P$ and, by taking the adjoint, $PD(g)P = PD(g)$. Implies that $1 - P$ also projects onto an invariant subspace. Basically, reducible implies upper triangular form, and taking the adjoint implies it's also lower triangular, so it must be block-diagonal.

⋆ Next time: Schur's lemmas.