Physics 220, Lecture 2

• Last time: left cosets gH of subgroup $H \subset G$, is the set of all gh for $h \in H$. Note that gH is not a group, it does not contain the identity element. Note also that it contains the same number of elements as H: $|gH| = |H|$, and none of its elements are in H. Suppose that we take a G element g' that is neither in H nor in gH, then the coset $g'H$ is also of order $|H|$, and none of its elements are in H or gH. Eventually, each and every G element fits into some coset, so we can decompose $G = H + g_1 H + \dots g_{m-1} H$. Likewise for right cosets. This shows that $|H| = |G|/m$ with m an integer.

Examples for S_3. Take $H_1 = \{e, a_1, a_2\}$ the Z_3 cyclic subgroup. Then $G = H_1 + a_3 H_1$, where the coset $a_3 H_1 = \{a_3, a_4, a_5\}$. Now consider H_2 one of the Z_2 subsets, say $H_2 = \{e, a_3\}$. Then $G = H_2 + a_4 H_2 + a_5 H_2$, with $a_4 H_2 = \{a_4, a_2\}$ and $a_5 H_2 = \{a_5, a_1\}$. Note that $a_3 H_1 = H_1 a_3$, but $a_4 H_2 \neq H_2 a_4$. For H_2, left and right cosets differ.

• Normal or invariant subgroups H: $gH = Hg$ for all $g \in G$. For example, the $Z_3 \subset S_3$ is normal, whereas the $Z_2 \subset S_3$ is not. Similarly, D_2 is a normal subgroup of S_4.

• Factor group G/H is a legit subgroup of G if (though perhaps not only if) H is normal. For example, $S_3/Z_3 = Z_2$.

• Conjugacy classes: sets S such that $g^{-1} S g = S$ for all $g \in G$. For abelian groups, each element is in its own conjugacy class. For every group e is in its own conjugacy class. S_3 example of conjugacy classes.

Normal subgroups contain complete conjugacy classes, e.g. $Z_3 \subset S_3$ and $D_2 \subset S_4$ (where $C_2 = (12)(34)$, $C_a = (13)(24)$, $C_b = (14)(23)$.)

• D_n and its conjugacy classes. Let's call the D_n elements C_n^k, for $k = 1, \dots n-$ (with $C_n^n = e$), and R_k. $D_3 \cong S_3$ has classes: $e; C_3, C_3^2; R_1, R_2, R_3$. D_4 has classes: $e; C_4, C_4^3; C_4^2; R_1, R_3; R_2, R_4$. Generally, $D_{n=2p}$ has e and C_{2p}^p each in their own class, the remaining n-fold rotations form $p-1$ classes with 2 elements in each class, and the 2-fold elements R_k form 2 classes of p elements each; the total is $(n+6)/2$ classes. For $D_{n=2p+1}$, all 2-fold elements R_k are in the same class, e forms a class, and the remaining n fold rotations give p classes with 2 elements each; the total is $(n+3)/2$ classes.

• General S_n. Write elements in terms of permutations, e.g. $e = (1)(2)\dots(n)$; general element as k_j j-cycles, with $\sum_{j=1}^n j k_j = n$, e.g. S_8 has an element $(123)(47)(568)$. Conjugacy class contains all elements with given choice of k_j, e.g. S_8 element is together with all $k_2 = 1$, $k_3 = 2$ elements. Some examples: in S_4, conjugating (23) by (12) gives (13); conjugating (234) by (12) gives (134) etc.

Such a class has $n!/\prod_j j^{k_j} k_j!$ elements.

Define $\lambda_j \equiv \sum_{i=j}^n k_j$, so $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\sum_{j=1}^n \lambda_j = n$. The different conjugacy classes are thus related to the different partitions of n. Draw diagrams with n boxes, with λ_1 in the first row, λ_2 in the second, etc.

S_4 example of conjugacy classes, with 1, 6, 3, 8, 6 elements.

• product groups: $G = H_1 \ldots H_n$ if elements $h_i \in H_i$ commute for $i \neq j$ and each $g \in G$ can uniquely be written as $g = h_1 \ldots h_n$. Follows that each H_i must be invariant subgroup of G. Example: $Z_6 = Z_2 \times Z_3$.