Physics 220, Lecture 19

$\star$ Reference: Georgi.

• Last time: Clebsch-Gordon decomposition, via Young tableaux. Put as in the top row and bs in the bottom. No two as or bs in same column and also, reading diagram Hebrew style, number of as must be greater or equal to number of bs. E.g. $(1,1)\times(1,1) = 8\times 8 = (2,2)+(3,0)+(0,3)+(1,1)+(1,1)+(0,0) = 27+10+\overline{10}+8+8+1$.

The two **8**s on the RHS means that there are two independent ways to make a singlet from three adjoints, A, B, C. These can be thought of as $f\sim \mathrm{Tr}(A[B,C])$ and $d\sim \mathrm{Tr}(A\{B,C\})$, e.g. $f^{abc}\sim \mathrm{Tr}(T^a[T^b,T^c])$, and $d^{abc}\sim \mathrm{Tr}(T^a\{T^b,T^c\})$.

• Back to $SU(3)_F$. Form baryons from 3 quarks, $3\times 3\times 3 = 1+8+8+10$. The 8 are the spin $j=1/2$ baryons, the P, N, and friends. The 10 are the $j=3/2$ baryons, the $\Delta^{++,+,0,-}$, $\Sigma^{*+,0,-}$, $\Xi^{*0,-}$, Ω^-. Recall $Q=T_3+Y/2$ and $Y=B+S=2T_8/\sqrt{3}$.

• Wigner-Eckart type analysis of mass splittings in the $SU(3)_F$ baryon multiplet,

$$B^i_j = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}} & \Sigma^+ & P \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}} & N \\ \Xi^- & \Xi^0 & -2\Lambda/\sqrt{6} \end{pmatrix}.$$

$H\approx H_{VS}+H_{MS}$, where H_{VS} commutes with $SU(3)_F$ and $H_{MS}=[T_8]^i_j O*_i$ transforms like the adjoint. Then $\langle B|H_{MS}|B\rangle$ involves two independent matrix elements, corresponding to the two $8s$ in 8×8. So $\langle B|H_{MS}|B\rangle = XTr(B^\dagger BT_8)+Y\mathrm{Tr}(B^\dagger T_8 B)$. Implies the Gell-Mann-Okubo formula, $2(M_N+M_\Xi)=3M_\Lambda+M_\Sigma$, which can be solved for M_Λ. Plugging in $M_N=940$, $M_\Sigma=1190$, $M_\Xi=1320MeV$, gave $M_\Lambda=1110MeV$, which was a prediction that turned out to be correct (less than 1% error).

• $SU(3)_C$ and quarks.

• General $SU(N)$. The simple roots, α_i with $\alpha_i\cdot\alpha_j=\delta_{ij}-\frac{1}{2}\delta_{i,j\pm 1}$. The fundamental **N** has weights given by N vectors, each $N-1$ dimensional, which can be written as ν_i with components $[\nu_i]_m=[H_m]_{ii}$; these vectors satisfy $\sum_{i=1}^N \nu_i=0$ and $\nu_i\cdot\nu_j=-\frac{1}{2N}+\frac{1}{2}\delta_{ij}$, $i=1\dots N$. The simple roots can be written in terms of these as $\alpha_i=\nu_i-\nu_{i-1}$. The fundamental weights are $\mu_j=\sum_{i=1}^j \nu_i$. The general irrep has highest weight $\mu=\sum_k q_k\mu_k$. This is represented by the Young tableau with q_k columns having k boxes. Again, the procedure is to first symmetrize in the rows and then antisymmetrize in the columns. Can also write the irrep as $[\ell_1,\ell_2,\ldots]$, where ℓ_i give the number of boxes in each column, e.g. the adjoint is $[N-1,1]$.

Find the dimension of the irrep by the factors over hooks rule, F/H. Recall for S_N the irreps can also be written as Young tableau, and there the dim of the irrep was $n!/H$.

• $SU(4)$ example, and relation to $SO(6)$.

• Approximate $SU(6)$ spin flavor symmetry. $SU(6)_{SF} \supset SU(3)_F \times SU(2)_S$ with $6 \to (3,2)$. The baryons fit into the $56 \to (10,4)+(8,2)$, corresponding to being completely symmetric objects in the spin and flavor indices of three quarks. The Pauli principle complete antisymmetry comes upon including $SU(3)_C$, since they are all color singlets.

The magnetic moment $eq\vec{\sigma}/2m$ is in the adjoint of $SU(6)$, so consider matrix elements $\langle 56|35|56\rangle$. Show there is a single 56 in 35×56. So the magnetic moments are all related to each other, all can be determined from any one in the approximation where $SU(6)$ works. Implies e.g. $\mu_P/\mu_N = -3/2$, which is pretty close: $\mu_P \approx 2.79(e/2m_p)$ and $\mu_N \approx -1.91(e/2m_p)$. Understand from the quark model, e.g. $\mu_p \approx 3$ from the quarks.

• $SU(3) \times SU(2) \times U(1)_Y$ representations of the quarks and leptons.

• $SU(5)$ and its $SU(3) \times SU(2) \times U(1)$ subgroup.

• For $SO(2r)$, the fundamental has weights given by $\pm e_k$, where e_k are unit vectors with components $[e_k]_m = \delta_{km}$. The roots are $\pm e_j \pm e_k$ with $k \neq j$, the positive roots are $e_j \pm e_k$ with $j < k$, and the simple roots are $e_j - e_{j+1}$ for $j = 1 \dots r-1$, and also $e_{r-1} + e_r$, where the last two are the ones at the end of the Dynkin diagram, both attaching to the previous one. For $SO(2r+1)$ the simple roots can also be written in terms of the same e_j, but with the last one, $e_{r-1} + e_r$, replaced with just e_r. The fundamental weights of $SO(2r+1)$ are $\mu_j = \sum_{k=1}^{j} e_k$ and $\mu_r = \frac{1}{2}\sum_{k=1}^{r} e_k$. The last one is the spinor. For $SO(2r)$ there are two spinors, $\mu_r = \frac{1}{2}(e_1 + \dots + e_r - e_{r+1})$ and $\mu_{r+1} = \frac{1}{2}(e_1 + \dots + e_r + e_{r+1})$. The spinors of $SO(2r+1)$ are real or pseudo-real. The spinors of $SO(2r)$ can be real, pseudo-real, or complex (e.g. $SO(8k+2)$ and $SO(8k+6)$ have complex spinors.

• $SO(10)$, vector, and spinors. Example of weights of the 16 spinor irrep. The $SU(5)$ subgroup.

• $SO(2r)$ and $SO(2r+1)$ and their $SU(r)$ subgroup. Start with Clifford algebra, $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$ for $i, j = 1 \dots N$ and form $M_{jk} = \frac{1}{4i}[\Gamma_j, \Gamma_k]$, which form a rep of $SO(N)$. Consider $A_j = \frac{1}{2}(\Gamma_{2j-1} - i\Gamma_{2j})$ and $A_j^\dagger = \frac{1}{2}(\Gamma_{2j-1} + i\Gamma_{2j})$, which satisfy $\{A_j, A_k^\dagger\} = \delta_{jk}$, and all other anticommutators vanish. Can form $SU(r)$ generators $T_a = \sum_{j,k} A_j^\dagger [T_a]_{jk} A_k$. Under this $SU(r)$ subgroup, the spinors are $\sum_{j=0}^{r}[2j]$ and $\sum_{j=0}^{r}[2j+1]$.