Physics 220, Lecture 18

$\star$ Reference: Georgi.

• Continue with $SU(3)$ and tensor methods. General $\mu_{max} = \ell_1\mu_1 + \ell_2\mu_2 = (\frac{1}{2}(\ell_1 + \ell_2), (\ell_1 - \ell_2)\frac{\sqrt{3}}{6})$. Draw weights, with $\ell_1 + 1$ on the side parallel to $\alpha_1$, and $\ell_2 + 1$ on the side parallel to $\alpha_2$. Degeneracy result: Going in from one layer to the next, the degeneracy of the weights in each layer increases by one each time until one reaches a triangular layer, after which the degeneracy remains constant. The total dimension of the $(\ell_1, \ell_2)$ irrep is $\frac{1}{2}(\ell_1 + 1)(\ell_2 + 1)(\ell_1 + \ell_2 + 2)$.

Write the **3** rep of $SU(3)$ as $|\frac{1}{2}, \sqrt{3}/6\rangle = |_1\rangle$, $|-\frac{1}{2}, \sqrt{3}/6\rangle = |_2\rangle$, and $|0, -1/\sqrt{3}\rangle = |_3\rangle$. The generators act on these as $T_a|_i\rangle = |_j\rangle[T_a]^j_i$. Likewise define the $\overline{\mathbf{3}}$ with generators $-T^*$, so $|^i\rangle$ states with $T_a|^i\rangle = -|^j\rangle[T_a]^i_j$. Now consider states $|v\rangle$ in the $(n, m)$ tensor product, with basis elements $|^{i_1\cdots i_m}_{j_1\cdots j_n}\rangle$. Invariant tensors, $\delta^i_j$, $\epsilon_{ijk}$, $\epsilon^{ijk}$. So irreps are the $(n, m)$ tensors with upper and lower indices each separately symmetrized, and will all traces subtraced out. Use this to get the dimension of the $(n, m)$ irrep, $D(n, m) = B(n, m) - B(n-1, m-1)$, with $B(n, m) = \begin{pmatrix} n+2 \\ 2 \end{pmatrix}\begin{pmatrix} m+2 \\ 2 \end{pmatrix}$.

Useful for understanding tensor products, e.g. $u^iv^j$ and $u^iv_j$.

• Young tableaux. Write $(n, m)$ irrep $A^{i_1\cdots i_n}_{j_1\cdots j_m}$ with all upper indices, raising each $j$ via $\epsilon^{jk\ell}$. Write as $m$ columns having 2 boxes, and $n$ with 1 box. Symmetrize in each row, then antisymmetrize in each column.

• Clebsch-Gordon decomposition, via Young tableaux. Put $a$s in the top row and $b$s in the bottom. No two $a$s or $b$s in same column and also, reading diagram Hebrew style, number of $a$s must be greater or equal to number of $b$s. E.g. $(1, 1) \times (1, 1) = 8 \times 8 = (2, 2) + (3, 0) + (0, 3) + (1, 1) + (1, 1) + (0, 0) = 27 + 10 + \overline{10} + 8 + 8 + 1$.

The two **8**s on the RHS means that there are two independent ways to make a singlet from three adjoints, $A$, $B$, $C$. These can be thought of as $f \sim \mathrm{Tr}(A[B, C])$ and $d \sim \mathrm{Tr}(A\{B, C\})$, e.g. $f^{abc} \sim \mathrm{Tr}(T^a[T^b, T^c])$, and $d^{abc} \sim \mathrm{Tr}(T^a\{T^b, T^c\})$.

• Back to $SU(3)_F$. Form baryons from 3 quarks, $3 \times 3 \times 3 = 1 + 8 + 8 + 10$. The 8 are the spin $j = 1/2$ baryons, the $P$, $N$, and friends. The 10 are the $j = 3/2$ baryons, the $\Delta^{++,+,0,-}$, $\Sigma^{*+,0,-}$, $\Xi^{*0,-}$, $\Omega^-$. Recall $Q = T_3 + Y/2$ and $Y = B + S = 2T_8/\sqrt{3}$.

• Wigner-Eckart type analysis of mass splittings in the $SU(3)_F$ baryon multiplet,

$$B^i_j = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & P \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & N \\ \Xi^- & \Xi^0 & -2\Lambda/\sqrt{6} \end{pmatrix}.$$

$H \approx H_{VS} + H_{MS}$, where $H_{VS}$ commutes with $SU(3)_F$ and $H_{MS} = [T_8]^i_j O*_i$ transforms like the adjoint. Then $\langle B|H_{MS}|B\rangle$ involves two independent matrix elements, corresponding to the two $8s$ in $8 \times 8$. So $\langle B|H_{MS}|B\rangle = XTr(B^\dagger BT_8) + Y\mathrm{Tr}(B^\dagger T_8 B)$. Implies the Gell-Mann-Okubo formula, $2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma$, which can be solved for $M_\Lambda$. Plugging in $M_N = 940$, $M_\Sigma = 1190$, $M_\Xi = 1320 MeV$, gave $M_\Lambda = 1110 MeV$, which was a prediction that turned out to be correct (less than 1% error).

• $SU(3)_C$ and quarks.

• General $SU(N)$. The simple roots, $\alpha_i$ with $\alpha_i \cdot \alpha_j = \delta_{ij} - \frac{1}{2}\delta_{i,j\pm 1}$. The fundamental **N** has weights given by $N$ vectors, each $N-1$ dimensional, which can be written as $\nu_i$ with $\nu_i \cdot \nu_j = -\frac{1}{2N} + \frac{1}{2}\delta_{ij}$. The simple roots can be written in terms of these as $\alpha_i = \nu_i = \nu_{i-1}$. The fundamental weights are $\mu_j = \sum_{i=1}^j \nu_i$. The general irrep has highest weight $\mu = \sum_k q_k \mu_k$. This is represented by the Young tableau with $q_k$ columns having $k$ boxes. Again, the procedure is to first symmetrize in the rows and then antisymmetrize in the columns.

Find the dimension of the irrep by the factors over hooks rule, $F/H$. Recall for $S_N$ the irreps can also be written as Young tableau, and there the dim of the irrep was $n!/H$.

• $SU(4)$ example, and relation to $SO(6)$.

• Approximate $SU(6)$ spin flavor symmetry. $SU(6)_{SF} \supset SU(3)_F \times SU(2)_S$ with $6 \to (3,2)$. The baryons fit into the $56 \to (10,4) + (8,2)$, corresponding to being completely symmetric objects in the spin and flavor indices of three quarks. The Pauli principle complete antisymmetry comes upon including $SU(3)_C$, since they are all color singlets.

The magnetic moment $eq\vec{\sigma}/2m$ is in the adjoint of $SU(6)$, so consider matrix elements $\langle 56|35|56\rangle$. Show there is a single 56 in $35 \times 56$. So the magnetic moments are all related to each other, all can be determined from any one in the approximation where $SU(6)$ works. Implies e.g. $\mu_P/\mu_N = -3/2$, which is pretty close: $\mu_P \approx 2.79(e/2m_p)$ and $\mu_N \approx -1.91(e/2m_p)$. Understand from the quark model, e.g. $\mu_p \approx 3$ from the quarks.