Physics 220, Lecture 17

⋆ Reference: Georgi chapters 8-9, a bit of 20.

• Last time: classify all simple, compact Lie algebras from their A_{ji}. Require 3 properties: (1) $\det A \neq 0$ (since the simple roots are linearly independent); (2) $A_{ji} < 0$ for $i \neq j$; (3) $A_{ij}A_{ji} = 0$,1, 2, 3. From these can prove many constraints. For example, taking $\alpha \equiv \sum_i \alpha_i/|\alpha_i|$ it follows from $\alpha^2 > 0$ that the number of joined nodes is strictly less than the number of nodes. Therefore, there can't be any closed loops. Draw all 2 allowed Dynkin diagrams for 3 nodes, following from fact that the sum of angles between any 3 linearly independent vectors must be $< 2\pi$. This same constraint applies to any subdiagram of a Dynkin diagrams. Another result following from this is that at most 3 lines can connect to any node. We can freely cut apart diagrams and shrink lines to get subsystems, which must satisfy same constraints. If a node γ connects with two single lines to two other nodes, α and β (with $\alpha \cdot \beta = 0$), then there is another allowed diagram where γ connects with a double node to $\alpha + \beta$. Draw several examples with non-linearly independent α_i, and corresponding μ_j such that $(\sum_j \mu_j \alpha_j)^2 = 0$ (note each μ_j is the average of those at connecting nodes). Draw all allowed Dynkin diagrams.

Extended Dynkin diagrams: include α_0, which is the lowest root, as an extra node in the diagram. Since it's lowest, $\alpha_0 - \alpha_j$ is not a root, so its $q = 0$ and thus A_{0j} and A_{j0} are non-positive integers. So satisfy same diagram rules as above, with the exception that there is a single linear relation among the nodes now. The extended Cartan matrix $\tilde{A}$ has rank r, so $\det \tilde{A} = 0$. Draw the extended Dynkin diagrams, with the corresponding $\alpha_0 = -\mu_{hr}$, where hr means highest root. So $-A_{j0} = 2(\alpha_j \cdot \mu_{hr})/\mu_{hr}^2$ and $-A_{0j} = 2(\alpha_j \cdot \mu_{hr})/\alpha_j^2$; as well soon mention, $-A_{0j}$ are thus the Dynkin coefficients of the highest root. These coefficients are enough to determine the linear relation of the diagram, which says how the highest root is written in terms of the other roots.

Draw extended Dynkin diagrams and give $\alpha_0 = -\mu_{max}$ for the adjoint representation.

The extended Dynkin diagram has another use: deleting any node gives a the Lie algebra's maximal regular subalgebras. A subalgebra is regular if its roots and Cartan generators are part of the original algebra. It's maximal if it has the same rank as the original algebra.

• Consider a highest weight μ of a general irrep., i.e. $E_{\alpha_j}|\mu\rangle$ for all simple roots α_j, i.e. all $p_j = 0$. So $2\alpha_j \cdot \mu/\alpha_j^2 = q_j \geq 0$. Here, q_j are called the Dynkin coefficients of the highest weight.

The fundamental weights, by definition, have $2\alpha_j \cdot \mu_i/\alpha_j^2 = \delta_{ij}$. So the general highest weight is $\mu = \sum_{j=1}^r \ell_j \mu_j$, with $\ell_j \geq 0$. For a highest weight, $\ell_j = q_j$, and for a general weight $\ell_j = q_j - p_j$ for the $SU(2)_j$ associated with simple root α_j. Start with the highest weight, and construct the full representation by successive lowering, stopping when $q_i > 0$ is no longer satisfied.

$SU(3)$ example: Taking $\alpha_{1,2} = (\frac{1}{2}, \pm\sqrt{3}/2)$, determine $\mu_{1,2} = (\frac{1}{2}, \pm\sqrt{3}/6)$. Equivalently, get $\mu_1 = (2\alpha_1 + \alpha_2)/3$ and $\mu_2 = (\alpha_1 + 2\alpha_2)/3$ from inverting the Cartan matrix.

A general $SU(3)$ irrep is then given by starting with $\mu_{max} = \ell_1\mu_1 + \ell_2\mu_2$, and filling out the rep by lowering with $E_{-\alpha_1}$ and $E_{-\alpha_2}$.

The $\mathbf{3}$ has $\mu_{max} = \mu_1$ and consists of μ_1, $\mu_1 - \alpha_1$, and $\mu_1 - \alpha_2$. The $\overline{\mathbf{3}}$ has $\mu_{max} = \mu_2$ and consists of μ_2, $\mu_2 - \alpha_2$, and $\mu_2 - \alpha_2 - \alpha_1$. The adjoint has $\mu_{max} = \mu_1 + \mu_2$. The $\mathbf{6}$ has $\mu_{max} = 2\mu_1$. The $\mathbf{10}$ has $\mu_{max} = 3\mu_1$.

• (Aside on Weyl reflections, in general. In the $SU(2)_\alpha$ collection of states $|\mu + p\alpha\rangle \ldots |\mu\rangle \ldots |\mu - q\alpha\rangle$, can reflect $L_z \to -L_z$. Corresponds to $\mu \to \mu - 2(\mu \cdot \alpha)\alpha/\alpha \cdot \alpha$, reflect weights in plane perpendicular to α.)

• Back to $SU(3)$. General $\mu_{max} = \ell_1\mu_1 + \ell_2\mu_2 = (\frac{1}{2}(\ell_1 + \ell_2), (\ell_1 - \ell_2)\frac{\sqrt{3}}{6})$. Draw weights, with $\ell_1 + 1$ on the side parallel to α_1, and $\ell_2 + 1$ on the side parallel to α_2.

Degeneracy result: Going in from one layer to the next, the degeneracy of the weights in each layer increases by one each time until one reaches a triangular layer, after which the degeneracy remains constant. The total dimension of the (ℓ_1, ℓ_2) irrep is $\frac{1}{2}(\ell_1+1)(\ell_2+1)(\ell_1 + \ell_2 + 2)$.

(Aside: for any group, the dimension of irrep with highest weight μ is given by the Weyl formula: $D(\mu) = \prod_{\alpha>0}(\mu + \rho, \alpha)/(\rho, \alpha)$, where $2\rho = \sum_{\alpha>0}\alpha$.)

• Tensor methods for $SU(3)$: write the $\mathbf{3}$ rep of $SU(3)$ as $|\frac{1}{2}, \sqrt{3}/6\rangle = |_1\rangle$, $|-\frac{1}{2}, \sqrt{3}/6\rangle = |_2\rangle$, and $|0, -1/\sqrt{3}\rangle = |_3\rangle$. The generators act on these as $T_a|_i\rangle = |_j\rangle[T_a]^j_i$. Likewise define the $\overline{\mathbf{3}}$ with generators $-T^*$, so $|^i\rangle$ states with $T_a|^i\rangle = -|^j\rangle[T_a]^i_j$. Now consider states $|v\rangle$ in the (n, m) tensor product, with basis elements $|^{i_1\cdots i_m}_{j_1\cdots j_n}\rangle$. Invariant tensors, δ^i_j, ϵ_{ijk}, ϵ^{ijk}. So irreps are the (n, m) tensors with upper and lower indices each separately symmetrized, and will all traces subtraced out. Use this to get the dimension of the (n, m) irrep, $D(n, m) = B(n, m) - B(n-1, m-1)$, with $B(n, m) = \begin{pmatrix} n+2 \\ 2 \end{pmatrix}\begin{pmatrix} m+2 \\ 2 \end{pmatrix}$.

Useful for understanding tensor products, e.g. $u^i v^j$ and $u^i v_j$.