Physics 220, Lecture 13

$\star$ Reference: Georgi chapters 4-5.

• Illustrate Wigner Eckart for $SU(2)$.

Operators transform under rotations, O^j_m with $[J_a, O^j_m] = O^j_{m'}[J^j_a]_{mm'}$.

Example: $\vec{r}$, write as $r_0 = r_3$, and $r_{\pm 1} = [J_\pm, r_0] = \mp(r_1 \pm i r_2)/\sqrt{2}$.

Wigner-Eckart accounts for how $O^{j_1}_{m_1}|j_2, m_2, \alpha\rangle$ transforms, exactly like the tensor product of representations:

$$\langle jm\beta|O^{j_1}_{m_1}|j_2 m_2, \alpha\rangle = \delta_{m,m_1+m_2}\langle jm, j_1 j_2|j_1 m_1, j_2 m_2\rangle\langle j\beta|O^{j_1}|j_2\alpha\rangle.$$

Example: consider the operators r_i for j_2 and $j = \frac{1}{2}$. If $\langle\frac{1}{2}, \frac{1}{2}, \alpha|r_3|\frac{1}{2}, \frac{1}{2}, \beta\rangle = A$, Wigner-Eckart relates this to $\langle\frac{1}{2}, -\frac{1}{2}, \alpha|r_-|\frac{1}{2}, \frac{1}{2}, \beta\rangle$ and $\langle\frac{1}{2}, \frac{1}{2}, \alpha|r_+|\frac{1}{2}, -\frac{1}{2}, \beta\rangle$: these all correspond to the $j = (1/2)$ on the RHS of $(1/2) \otimes (1) = (1/2) \oplus (3/2)$ (the $(3/2)$ would correspond to another, independent matrix element coefficient). To get the relation use e.g. $|3/2, 3/2\rangle = r_+|\frac{1}{2}, \frac{1}{2}, \beta\rangle$ and $|3/2, \frac{1}{2}\rangle = \sqrt{\frac{2}{3}}J^-|3/2, 3/2\rangle = \sqrt{\frac{2}{3}}r_0|\frac{1}{2}, \frac{1}{2}, \beta\rangle + \frac{1}{\sqrt{3}}r_+|\frac{1}{2}, -\frac{1}{2}, \beta\rangle$ and $|\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}}r_0|\frac{1}{2}, \frac{1}{2}, \beta\rangle - \sqrt{\frac{2}{3}}r_+|\frac{1}{2}, -\frac{1}{2}, \beta$. Using $0 = \langle\frac{1}{2}\frac{1}{2}\alpha|\frac{3}{2}\frac{1}{2}\rangle$ see e.g. $\langle\frac{1}{2}, \frac{1}{2}, \alpha|r_+|\frac{1}{2}, -\frac{1}{2}, \beta\rangle = -\sqrt{2}A$. Conclude e.g. $\langle\frac{1}{2}, \frac{1}{2}, \alpha|r_1|\frac{1}{2}, -\frac{1}{2}\beta\rangle = A$.

• Isospin: $N = \begin{pmatrix} p \\ n \end{pmatrix}$, created by $a^\dagger_{N,I,\alpha}$, with $I = \pm\frac{1}{2}$. These are fermionic operators so they anticommute. In particular, the bound state of two must be net antisymmetric in spin and isospin labels. Quarks $\begin{pmatrix} u \\ d \end{pmatrix}$ likewise form an isospin doublet. States are antisymmetric under interchange, so e.g. acting with two creation operators on the vacuum gives either $I = 0$, $s = 1$ (deuteron, ρ vector meson) or $I = 1$, $s = 0$ (pions), since quarks $\begin{pmatrix} u \\ d \end{pmatrix}$ form an isospin doublet. Combine 3 quarks to get $I = \frac{1}{2}$ (i.e. $\begin{pmatrix} p \\ n \end{pmatrix}$) or $I = 3/2$ (the Δ).

• Isospin application examples: $N + N \to D + \pi$, $\pi + N \to \pi + N$.