Physics 220, Lecture 11

⋆ Reference: Georgi chapters 3-6.

• Recall: $D_r(g(\alpha)) = e^{i\alpha_A T_r^A}$, where the Lie Algebra generators satisfy $[T^A, T^B] = if_{ABC}T^C$. Construct irreps of group via irreps of algebra. These are found by the highest weight procedure, which we discussed last time for the case of $SU(2)$: $[T_1, T_2] = iT_3$. The basis for the irreps are $|jm\alpha\rangle$, where $j = \ell/2$ labels the *highest weight*, i.e. the highest T_3 eigenvalue of the irrep, and $T_3|jm\alpha\rangle = m|jm\alpha\rangle$ and $T_\pm|jm\alpha\rangle = \sqrt{(j \pm m + 1)(j \mp m)/2}|j, m \pm 1\alpha\rangle$, with $T_\pm \equiv (T_1 \pm iT_2)/\sqrt{2}$. For $SU(2)$, have $T_a = J_a$, angular momentum.

We know from angular momentum that our irreps also diagonalize $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, which satisfies $[\vec{J}^2, J_a] = 0$. This is an example of a (quadratic) Casimir, a product of generators which commutes with all generators. As mentioned last time, the rank of the group is the dimension of the Cartan subalgebra. As we'll see, the number of Casimirs equals the rank of the group, e.g. for $SU(N)$ there are Casimirs of degree $2 \ldots N$.

• Draw pictures of $SU(2)$ irrep weight diagrams.

• Group elements act on the basis as $U(g)|jm\rangle = |jm'\rangle D^j(g)_{m'm}$, where the index order ensures $U(g_1)U(g_2) = D(g_1 g_2)$. As discussed last time, for $SU(2)$ or $SO(3)$ it is conventional to write $D(g) = R(\alpha, \beta, \gamma) = R_3(\alpha)R_2(\beta)R_3(\gamma)$, and e.g.

$$D^{j=1/2}(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}.$$

Recall that $\langle\theta\phi|jm\rangle = Y_{jm}(\theta\phi)$. So $\langle\theta'\phi'| = \langle\theta\phi|U^{-1}(g)$ and in particular $\langle\theta\phi| = \langle 00|U(g^{-1})$, where $g(\alpha) = R(\phi, \theta, 0)$. Hence $Y_{\ell,m}(\theta, \phi) = \langle 00|U(g^{-1})|\ell m\rangle = \langle 00|\ell m'\rangle D^\ell(\theta, \phi, 0)_{mm'}$ and so $Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} D^\ell(\theta, \phi, 0)^*_{m0}$. Indeed, $|\theta\phi\rangle = \sum_{\ell m} D^\ell(\theta, \phi, 0)_{m0}\sqrt{\frac{2\ell+1}{4\pi}}$.

• Recall for finite groups

$$\sum_{g\in G} D^j(g^{-1})|jm\rangle\langle j'm'|D^{j'}(g) = \frac{|G|}{|r_j|}\delta_{jj'}\delta_{mm'}I.$$

The corresponding property here is

$$(2j + 1)\int dV_g D_j^\dagger(g)_{mn} D_{j'}(g)_{n'm'} = \delta_{jj'}\delta_{mm'}\delta_{nn'}.$$

There is also a completeness property (Peter-Weyl):

$$\sum_{jmn} (2j + 1)D_j(g)_{mn} D_j^\dagger(g')(nm) = \delta(g - g').$$

Here dV_g is the invariant measure (Haar), $dV_g = dV_{gh}$, e.g. $dV_g \propto d\alpha d(\cos\beta)d\gamma$, and $\delta(g-g')$ is correspondingly defined.

- Tensor products: $|ix\rangle = |i\rangle \otimes |x\rangle$, $[J_1^{1\otimes 2}]_{jyix} = [J_a^1]_{ji}\delta_{xy} + \delta_{ij}[J_a^2]_{xy}$.

  Clebsch-Gordon coefficients:

$$|jm, j_1 j_2\rangle = \sum_{m_1}\sum_{m_2} |j_1 m_1, j_2 m_2\rangle\langle j_1 m_1, j_2 m_2|jm, j_1 j_2\rangle.$$

Examples ($\frac{1}{2}\otimes\frac{1}{2} = 0 \oplus 1,\ \frac{1}{2}\otimes 1 = \frac{3}{2}\oplus\frac{1}{2}$.).