Physics 220, Lecture 10

⋆ Reference: Georgi chapters 2-5.

• Last time: $g(\alpha) = e^{i\alpha_A T^A}$, and group multiplication requires

$$[T^A, T^B] = i f_{ABC} T^C,$$

and then $\gamma_C = \alpha_A \beta_B f_{ABC}$. If there is a unitary rep, the T^A are hermitian and then it follows that the f_{ABC} **are real**, since $[T^A, T^B]^\dagger = [T^B, T^A]$. Can normalize by $Tr(T_a T_b) = k_a \delta_{ab}$, where $k_a > 0$ for compact Lie algebras, and can take all $k_a = \lambda$. Then $f_{abc} = -i\lambda^{-1}\mathrm{Tr}([T^a, T^b]T^c)$ is completely antisymmetric in the 3 indices. Simplest case: $f_{ABC} = \epsilon_{ABC}$, with $A = 1 \dots 3$.

Georgi: fun with exponentials, e.g. $\partial_{\alpha_b} e^{i\alpha_a X^a} = \int_0^1 ds e^{is\alpha_a X_a}(iX_b)e^{i(1-s)\alpha_c X_c}$.

• Adjoint representation: $(T_a)_{bc} = -i f_{abc}$. Since f_{abc} is completely antisymmetric and real, T_a in the adjoint representation are hermitian, showing that the adjoint rep is unitary.

• $SO(3)$, $O(3)$, and $SU(2)$: $f_{abc} = \epsilon_{abc}$; $T_a = J_a$ angular momentum ($\hbar = 1$).

$SO(3)$ rotations: $R_{\widehat{n}}(\psi)$, with $R_{\widehat{n}}(\pi) = R_{-\widehat{n}}(\pi)$: ball with antipodal points on boundary identified. $R(\alpha, \beta, \gamma) = R_3(\alpha)R_2(\beta)R_3(\gamma)$. In $SU(2)$:

$$R_{j=1/2}(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{-i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}.$$

The Cartan subalgebra is the subalgebra of commuting generators. Here it is one dimensional, generated by say J_3. The basis of a rep can be chosen to be eigenstates of the Cartan, and let j label the highest eigenvalue, $J_3|j, \alpha\rangle = j|j, \alpha\rangle$. Using $J^\pm = (J_1 \pm iJ_2)/\sqrt{2}$ and $[J_3, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = J_3$, then J_+ must annihilate the state with highest J_3 eigenvalue. Since J_- lowers the eigenvalue, $J_\pm|m, \alpha\rangle = N_m|m-1, \alpha\rangle$; also $J^+|m-1, \alpha\rangle = N_m|m, \alpha\rangle$. Get $N_j = \sqrt{j}$ and $N_m^2 = N_{m+1}^2 + n$. Solution is $N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$. Since lowering must eventually also stop giving new states, j is integer or half-integer. Write $|jm\rangle$, with $J_\pm|jm\rangle = \sqrt{(j \pm m + 1)(j \mp m)/2}|j, m \pm 1\rangle$.

For $j = \frac{1}{2}$, $J_a = \frac{1}{2}\sigma_a$; fundamental representation. For $j = 1$ (adjoint rep),

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$