6/5/09 Lecture outline

⋆ Reading: Zwiebach chapter 14 and 17.

• Summarize from last two lectures. The relativistic open string spectrum is given by

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(a_n^{I\dagger})^{\lambda_{n,I}}|p^\mu\rangle \quad \text{with} \quad M^2 = -p^2 = (N_\perp - 1)/\alpha', \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\lambda_{n,I}. \tag{1}$$

Where consistency requires $D = 26$.

The closed string is like a tensor product of two copies of the open string, corresponding to the left movers and right movers. In particular, the closed string states are

$$\begin{aligned} |\lambda, \widetilde{\lambda}\rangle &= [\prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(a_n^{I\dagger})^{\lambda_{n,I}}][\prod_{n=1}^{\infty}\prod_{I=2}^{D-1}(\widetilde{a}_n^{I\dagger})^{\widetilde{\lambda}_{n,I}}]|p^\mu\rangle \\ M^2 &= -p^2 = 2(N_\perp + \widetilde{N}_\perp - 2)/\alpha', \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\lambda_{n,I}, \quad N^\perp = \sum_{n=1}^{\infty}\sum_{I=1}^{D-1} n\widetilde{\lambda}_{n,I}, \end{aligned} \tag{2}$$

where there is a requirement that $N^\perp = \widetilde{N}^\perp$ to have $\sigma$ translation invariance.

• Let's count the states by defining $f(x) = \mathrm{Tr}_{states} x^{\alpha' M^2}$. Find

$$f_{os}(x) = x^{-1}\prod_{n=1}^{\infty}\frac{1}{(1-x^n)^{24}}$$

where we set $D - 2 = 24$. Similarly, for the closed string case, we have

$$f_{closed}(x, \bar{x}) = f_{os}(x) f_{os}(\bar{x}),$$

where we need to project out those states with different powers of $x$ and $\bar{x}$.

• Now consider the superstrings. The bosonic string has fields $X^I(\tau, \sigma)$, which are $D-2$ worldsheet scalars. Now we introduce $D-2$ worldsheet fermions

$$\Psi_R(\tau - \sigma)^I, \qquad \Psi_L^I(\tau + \sigma).$$

Here $R$ and $L$ are for right and left moving, and $I = 2 \ldots D-2$ (spacetime vector indices). There are two choices of boundary conditions for left movers, and similarly two choices for right movers:

$$\Psi^I(\tau, \sigma + 2\pi) = \pm\Psi^I(\tau, \sigma), \qquad + : \text{Ramond}, \quad - : \text{Nevu-Schwarz}.$$

In the NS sector we have

$$\Psi^I_{NS} \sim \sum_{n=-\infty}^{\infty} b^I_{n+\frac{1}{2}} e^{-i(n+\frac{1}{2})(\tau-\sigma)}.$$

In the R sector we have

$$\Psi^I_R \sim \sum_{n=-\infty}^{\infty} d^P_n e^{-in(\tau-\sigma)}.$$

The modes satisfy

$$\{b^I_r, b^J_s\} = \delta_{r+s,0}\delta^{IJ}, \qquad \{d^I_n, d^J_m\} = \delta_{n+m,0}\delta^{IJ},$$

where $\{A, B\} \equiv AB + BA$ is the anti-commutator, reflecting the fermionic nature of the modes.

The NS sector states are

$$|\lambda, \rho\rangle_{NS} = \prod_{I=2}^{D-2} (a^{I\dagger}_n)^{\lambda_{n,I}} \prod_{J=2}^{D-1} \prod_{r=\frac{1}{2},\frac{3}{2}\cdots} (b^J_{-r})^{\rho_{r,J}} |NS\rangle \otimes |p\rangle,$$

where the $\rho_{r,J}$ are either zero or one (Fermi statistics).

The R sector states are

$$|\lambda, \rho\rangle_R = \prod_{I=2}^{D-2} \prod_n (a^{I\dagger}_n)^{\lambda_{n,I}} \prod_{J=2}^{D-1} \prod_{m=1}^{\infty} (d^J_{-m})^{\rho_{m,J}} |R_A\rangle \otimes |p\rangle,$$

Here $|R_A\rangle$ are the Ramond ground states, which are complicated thanks to the zero modes $d^I_0$. We take half of them $\frac{1}{2}(D-2)$ to be creation operators and the other half to annihilate the vacuum. So then there are $2^{\frac{1}{2}(D-2)}$ degenerate states. These form two equal groups, depending on whether there are an even number of creation operators, or an odd number. The former is called the $R-$ sector and labeled by $|R_a\rangle$, and the latter is called the $R+$ sector and labeled by $|R_{\bar{a}}\rangle$. The $\pm$ refer to worldsheet fermion number $(-1)^F$, where the vacuum has fermion number $(-1)^F = -1$.

- The mass-squared operator in the NS sector before normal ordering is

$$\alpha' M^2 = \tfrac{1}{2} \sum_{p\neq 0} \alpha^I_{-p}\alpha^I_p + \tfrac{1}{2} \sum_{r=n+\frac{1}{2}} r b^I_{-r} b^I_r.$$

Re-ordering, we have

$$\alpha' M^2 = N^\perp + \tfrac{1}{2}(D-2)(-\frac{1}{12} - \frac{1}{24}),$$

where the $-1/12$ was seen in the bosonic case, and the $-1/24$ is the analog coming from reordering the $b_r$. As in the bosonic case, the commutator $[M^{-I}, M^{-J}] = 0$ determines the spacetime dimension, here to be $D = 10$. So in the NS sector the mass squared operator is

$$\alpha' M^2 = -\tfrac{1}{2} + N^\perp, \qquad N^\perp = \sum_{p=1}^{\infty} p a_p^{\dagger I} a_p^I + \sum_{r=\frac{1}{2},\frac{3}{2},\dots} r b_{-r}^I b_r^I.$$

Similarly, in the R-sector, we have

$$\alpha' M^2 = \tfrac{1}{2} \sum_{p \neq 0} \alpha_{-p}^I \alpha_p^I + \tfrac{1}{2} \sum_m m d_{-m}^I d_m^I.$$

Re-ordering we have $\alpha' M^2 = N^\perp + \frac{1}{2}(D-2)(-\frac{1}{12} + \frac{1}{12})$, and the constants cancel, so

$$\alpha' M^2 = N^\perp, \qquad N^\perp = \sum_{p=1}^{\infty} p a_p^{\dagger I} a_p^I + \sum_{m=1}^{\infty} m d_{-m}^I d_m^I.$$

In particular, the Ramond ground states are massless.

- The NS spectrum generating function is

$$f_{NS}(x) = \frac{1}{\sqrt{x}} \prod_{n=1}^{\infty} \left( \frac{1 + x^{n-\frac{1}{2}}}{1 - x^n} \right)^8.$$

The R sector spectrum generating function is

$$f_{R\pm}(x) = 8 \prod_{n=1}^{\infty} \left( \frac{1 + x^n}{1 - x^n} \right)^8$$

where 8 accounts for the ground state degeneracy associated with $d_0^I$, in either the $R_+$ or the $R_-$ sector. We should also GSO project the NS sector, i.e. throw away states with $(-1)^F = -1$ to get the $NS+$ states, with generating function

$$f_{NS+}(x) = \frac{1}{2\sqrt{x}} \left[ \prod_{n=1}^{\infty} \left( \frac{1 + x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 - \left( \frac{1 - x^{n-\frac{1}{2}}}{1 - x^n} \right)^8 \right].$$

This projects out the tachyon – nice! Moreover, the states in $f_{R\pm}$ are spacetime fermions, whereas those in $f_{NS,+}$ are spacetime bosons, and their spectrum is degenerate, thanks to the identity $f_{R\pm}(x) = f_{NS+}(x)$ (which was proven as a mathematical identity around 150 years before the superstring was even first thought of!).

• For closed superstrings we can take the $NS+$ sector for both left and right movers, and the $R-$ sector for both left and right movers; this is the IIB superstring. Or we could take the $NS+$ sector for both left and right movers, and the $R-$ sector for left movers and the $R+$ sector for right movers; this is the IIA superstring.

The massless (NS+, NS+) states for both of these string theories consist of

$$\widetilde{b}^I_{-\frac{1}{2}}|NS\rangle_L \otimes b^J_{-\frac{1}{2}}|NS\rangle_R \otimes |p\rangle.$$

As in the bosonic case, these correspond to $g_{\mu\nu}$,, $B_{\mu\nu}$, $\phi$.

• Consider the closed, bosonic string on a circle, $X_{25} \sim X_{25} + 2\pi R$. If we were dealing with particles rather than strings, we know what would happen: the momentum in the circle direction is quantized (by $\psi \sim e^{ip\cdot x}$ being set equal to itself when going around the circle) as

$$p_{25} = \frac{n}{R}, \qquad n = 0, \pm 1, \pm 2 \ldots.$$

For a big circle, these are closely spaced together, and for a small circle they are widely separated. That's why it's hard to experimentally rule out the absence of tiny, rolled up, extra dimensions: it could just take more energy than we can make presently to excite one of the $n \neq 0$ "Kaluza-Klein modes."

Now we're going to describe something bizarre about strings: there is a symmetry, called T-dualtiy, which makes the physics invariant under $R \leftrightarrow \alpha'/R$. This is strange: a very big circle is physically indistinguishable from a very small circle! The reason is that, in addition to momentum, there are string winding modes, and T-duality exchanges them. For a big circle, the momentum modes are light and the winding modes are heavy, and for a tiny circle they're reversed, but same physics. Smallest possible effective distance, $R = \sqrt{\alpha'}$.

The winding number is given by $X(\tau, \sigma + 2\pi) - X(\tau, \sigma) = m(2\pi R)$. We then have $X = X_L + X_R$ with

$$X_L(\tau + \sigma) = const. + \tfrac{1}{2}\alpha'(p + w)(\tau + \sigma) + \text{oscillators},$$

$$X_R(\tau - \sigma) = const + \tfrac{1}{2}\alpha'(p - w)(\tau - \sigma) + \text{oscillators}.$$

Here

$$p = \frac{n}{R}, \qquad w = \frac{mR}{\alpha'}.$$

The T-duality symmetry comes from the symmetry $(p_L, p_R) \to (p_L, -p_R)$, where

$$p_L = \frac{n}{R} + \frac{mR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{mR}{\alpha'}.$$

Also, to have $X(\tau, \sigma + 2\pi) \sim X(\tau, \sigma) + 2\pi Rm$, we need $N^\perp - \widetilde{N}^\perp = nm$.