5/29/09 Lecture outline

⋆ Reading: Zwiebach chapter 12 and 13.

• Recall we were looking at the open string, in light cone gauge. So we fixed X^+ to be simply related to τ, found that the X^I are given by simple harmonic oscillators, and X^- is something complicated, which is where we left off.

• $X^+(\tau\sigma) = 2\alpha' p^+\tau = \sqrt{2\alpha'}\alpha_0^+\tau$. For X^- recall expansion, with $\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp$, where $L_n^\perp \equiv \frac{1}{2}\sum_p \alpha_{n-p}^I\alpha_p^I$ is the transverse Virasoro operator. There is an ordering ambiguity here, only for $L_0^\perp$:

$$L_0^\perp = \tfrac{1}{2}\alpha_0\alpha_0 + \tfrac{1}{2}\sum_{p=1}^\infty \alpha_{-p}^I\alpha_p^I + \tfrac{1}{2}\sum_{p=1}^\infty \alpha_p^I\alpha_{-p}^I.$$

The ordering in the last terms need to be fixed, so the annihilation operator α_p is on the right, using $\alpha_p^I\alpha_{-p}^I = \alpha_{-p}^I\alpha_p^I + [\alpha_p^I, \alpha_{-p}^I]$, which gives

$$L_0^\perp = \alpha' p^I p^I + \sum_{n=1}^\infty n a_n^{I\dagger} a_n^I,$$

where the normal ordering constant has been put into

$$2\alpha' p^- = \frac{1}{p^+}(L_0^\perp + a), \qquad a = \tfrac{1}{2}(D-2)\sum_{p=1}^\infty p.$$

This leads to

$$M^2 = \frac{1}{\alpha'}(a + \sum_{n=1}^\infty n a_n^{I\dagger} a_n^I).$$

The divergent sum for a is regulated by using $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ and analytically continuing to get $\zeta(-1) = -1/12$. So

$$a = -\frac{1}{24}(D-2).$$

• Virasoro generators and algebra (corresponds to worldsheet energy-momentum tensor). Since $(\alpha_n^I)^\dagger = \alpha_{-n}^I$, get $L_n^{\perp\dagger} = L_{-n}^\perp$. Also,

$$[L_m^\perp, \alpha_n^I] = -n\alpha_{n+m}^I.$$

$$[L_m^\perp, L_n^\perp] = (m-n)L_{n+m}^\perp + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0}.$$

• Spacetime Lorentz symmetry corresponds to conserved currents on worldsheet, with conserved charges

$$M_{\mu\nu} = \int_0^\pi (X_\mu \mathcal{P}^\tau_\nu - (\mu \leftrightarrow \nu)) d\sigma.$$

Plug in $\mathcal{P}^\tau_\nu = \frac{1}{2\alpha'}\dot{X}^\mu$ and plug in oscillator expansion of X^μ to get

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n).$$

In light cone gauge, have to be careful with M^{-I}, since X^- is constrained to something complicated,

$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma, \qquad \sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp.$$

and also careful to ensure that $[M^{-I}, M^{-J}] = 0$. Find, after appropriately ordering terms,

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I(L_0^\perp + a) + (L_0^\perp + a)x_0^I\right) - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^\infty \frac{1}{n}(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp).$$

Then get

$$[M^{-I}, M^{-J}] = -\frac{1}{\alpha' p^{+2}}\sum_{m=1}^\infty (\alpha^I_{-m}\alpha^J_m - (I\leftrightarrow J))[m(1-((D-2)/24)) + m^{-1}(((D-2)/24)+a)].$$

Since this must be zero, get $D = 26$ and $a = -1$.

The worldsheet Hamiltonian is thus

$$H = 2\alpha' p^+ p^- = L_0^\perp - 1.$$

• The states are obtained as

$$|\lambda\rangle = \prod_{n=1}^\infty \prod_{I=2}^{25} (a_n^{I\dagger})^{\lambda_{n,I}} |p^+, \vec{p}_T\rangle.$$

These states are eigenstates of

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \qquad N^\perp \equiv \sum_{n=1}^\infty n a_n^{I\dagger} a_n^I,$$

with eigenvalues

$$M^2 = \frac{1}{\alpha'}(-1 + N^\perp), \qquad N^\perp = \sum_n \sum_I n\lambda_{n,I}.$$

The groundstate is tachyonic (!). The first excited state is a massless spacetime vector with $D-2$ polarizations, i.e. a massless gauge field, like the photon (but in $D=26$)!

The tachyon is related to the fact that the D25 brane is unstable, it decays to the closed string vacuum. The closed bosonic string is also unstable, as we'll see next time. These instabilities can be cured by adding fermions and considering the superstring. Then the critical spacetime dimension is $D=10$.

The eigenstates satisfy the worldsheet SE:

$$i\frac{\partial}{\partial\tau}|\lambda\rangle = H|\lambda\rangle = (L_0^\perp - 1)|\lambda\rangle.$$

Writing $x^+ = 2\alpha' p^+\tau$, this becomes

$$(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(p^Ip^I + M^2))\phi_\lambda(x^+, p^+, \tau),$$

which is the KG (or generalization) wave equation for the corresponding field in spacetime.

• Now consider closed string case. Recall gauge conditions $n\cdot X = \alpha'(n\cdot p)\tau$, $n\cdot p = 2\pi n\cdot \mathcal{P}^\tau$, which yielded the constraints $(\dot X \pm X')^2 = 0$ and then the EOM were simply $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$. For the closed string, this means that $X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)$. The general solutions can then be written as

$$X_R^\mu(v) = \tfrac{1}{2}x_0^{L\mu} + \sqrt{\tfrac{1}{2}\alpha'}\alpha_0^\mu v + i\sqrt{\tfrac{1}{2}\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-inv},$$

and a similar expression for X_L^μ, with modes $\widetilde\alpha_n^\mu$. Since $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$, $\widetilde\alpha_0^\mu = \alpha_0^\mu$. Computing $\mathcal{P}^{\mu\mu} = \dot X^\mu/2\pi\alpha'$ then yields $\alpha_0^\mu = \sqrt{\tfrac{1}{2}\alpha'}p^\mu$.

The theory is quantized by taking $[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')] = -\delta(\sigma-\sigma')\eta^{IJ}$, which implies that

$$[\alpha_m^I, \alpha_n^J] = m\delta_{m+n,0}\eta^{IJ}, \qquad [\widetilde\alpha_m^I, \widetilde\alpha_n^J] = m\delta_{m+n,0}\eta^{IJ}$$

with no commutator between the left and right movers. It's now fairly similar to the open string case, but with the two sets of decoupled oscillators for the left and right movers. We define

$$(\dot X^I + X'^I)^2 \equiv 4\alpha'\sum_n \widetilde L_n^\perp e^{-in(\tau+\sigma)},$$

and a similar expansion for $(\dot X^I - X'^I)^2$ and $L_n^\perp$, involving $\tau-\sigma$. Then $L_n^\perp = \frac{1}{2}\sum_p \alpha_p^I\alpha_{n-p}^I$, and $L_0^\perp = \frac{\alpha'}{4}p^Ip^I + N^\perp$. The X^- are given in terms of these much as in the open string case, $\sqrt{2\alpha'}\alpha_n^- = 2L_n^\perp/p^+$, with a similar expression for the left movers. The worldsheet Hamiltonian is $H = L_0^\perp + \widetilde{L}_0^\perp - 2$ and $M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{2}{\alpha'}(N^\perp + \widetilde{N}^\perp - 2)$.

The closed string states are given by acting with left and right moving creation operators on $|p^+, p^I\rangle$, with the constraint that $N^\perp = \widetilde{N}^\perp$ (because of translation symmetry in shifting σ). The state with $N^\perp = 0$ is the bosonic closed string tachyon. Those with $N^\perp = 1$ are given by a $(D-2)^2$ matrix of indices in the transverse directions, and these are massless. The symmetric traceless part is the graviton, the antisymmetric tensor is a gauge field $B_{\mu\nu}$ which is an analog of A_μ, and the trace part is ϕ, called the "dilaton."

• The D-dimensional Newton's constant is given by $G^{(D)} = \ell_P^{D-2} \sim g^2(\alpha')^{(D-2)/2}$. Get $g \sim e^\phi$.