5/1/09 Lecture outline

⋆ Reading: Zwiebach chapter 7, 8, 9.

• Last time: the string generalization of $S = -mc\int ds$ is

$$S_{Nambu-Goto} = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

where we define $\dot{X}^\mu \equiv \frac{dx^\mu}{d\tau}$ and $X\mu' \equiv \frac{\partial X^\mu}{\partial \sigma}$ annd T_0 is the string tension, with $[T_0] = [F] = [ML/T^2]$.

The action is reparameterization invariant: can take $(\tau,\sigma) \to (\tau'(\tau,\sigma,\sigma'(\tau\sigma)$ and get $S \to S$. Enormous symmetry/redundancy in choice of (τ,σ); can "fix the gauge" to some convenient choice, and the physics is completely independent of the choice.

• We can write S_{NG} in terms of the Lagrangian density

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

For the open string, $\delta S = 0$ also requires $\int d\tau [\delta X^\mu P^\sigma_\mu]_0^{\sigma_0} = 0$, which requires for each μ index either of the Dirichlet or Neumann BCs, at each end:

$$\text{Dirichlet} \qquad \frac{\partial X^\mu}{\partial \tau}(\tau,\sigma_*) = 0 \qquad \to \qquad \delta X^\mu(\tau,\sigma_*) = 0,$$

$$\text{Neumann} \qquad \mathcal{P}^\sigma_\mu(\tau,\sigma_*) = 0.$$

• Exploit $(\tau,\sigma) \to (\tau',\sigma')$ reparameterization invariance to pick useful "gauges", to simplify the above equations. We will discuss choices such that we can impose constraints

$$\dot{X}\cdot X' = 0 \qquad \dot{X}^2 + X'^2 = 0. \tag{1}$$

In this case, we have

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X^{\mu\prime}, \tag{2}$$

and then the EOM is simply a wave equation:

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0. \tag{3}$$

Now let's explain these things in more detail.

• Static gauge: pick $\tau = t$. Verify sign inside $\sqrt{\cdot}$ in this case.

• In static gauge, there is no KE, so $L = -V$, and verify that string stretched length a, e.g. $X^1 = f(\sigma)$, has $V = T_0 a$. So $\mu_0 = T_0/c^2$.

• In static gauge, express S in terms of $\vec{v}_\perp = \partial_t\vec{X} - (\partial_t\vec{S}\cdot\partial_s\vec{X})\partial_s\vec{X}$ (with $ds \equiv |d\vec{X}|_{t=const} = |\partial_\sigma\vec{X}||d\sigma|$) to get $L = -T_0\int ds\sqrt{1-v_\perp^2/c^2}$. Also get

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\frac{(\partial_s\vec{X}\cdot\partial_t\vec{X})\dot{X}^\mu + (c^2 - (\partial_t\vec{X})^2)\partial_s X^\mu}{\sqrt{1-v_\perp^2/c^2}},$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{ds}{d\sigma}\frac{\dot{X}^\mu - (\partial_s\vec{X}\cdot\partial_t\vec{X})\partial_s X^\mu}{\sqrt{1-v_\perp^2/c^2}}.$$

• Show that endpoints move transversely, $\partial_s\vec{X}\cdot\partial_t\vec{X} = 0$, and at the speed of light, $v = c$, for the free (Neuman) BCs, using fact that $\mathcal{P}^{\sigma\mu} = 0$.

• Choose σ parameterization such that

$$\partial_\sigma\vec{X}\cdot\partial_\tau\vec{X} = 0 \qquad \text{and} \qquad d\sigma = \frac{ds}{\sqrt{1-v_\perp^2/c^2}} = \frac{dE}{T_0}.$$

(Using $H = \int T_0 ds/\sqrt{1-v_\perp^2/c^2}$ and $\partial_t(ds/\sqrt{1-v_\perp^2/c^2}) = 0$.) The last equation above is equivalent to $(\partial_\sigma\vec{X})^2 + c^{-2}(\partial_t\vec{X})^2 = 1$. With this worldsheet gauge choice,

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\partial_t X^\mu = \frac{T^0}{c^2}(c, \vec{v}_\perp), \qquad \mathcal{P}^{\sigma,\mu} = -T_0\partial_\sigma X^\mu = (0, -T_0\partial_\sigma\vec{X}).$$

The equation of motion is then simply $(\partial_t^2 - c^2\partial_\sigma^2)\vec{X} = 0$.

• Solution of the EOM for open string with free BCs: $\vec{X}(t,\sigma) = \frac{1}{2}(\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma))$ where the open string has $\sigma \in [0,\sigma_1]$ and $|\frac{d\vec{F}(u)}{du}|^2 = 1$ and $\vec{F}(u+2\sigma_1) = \vec{F}(u) + 2\sigma_1\vec{v}_0/c$. Example from book: $\vec{X}(t,\sigma=0) = \frac{\ell}{2}(\cos\omega t, \sin\omega t)$. Find $\vec{F}(u) = \frac{\sigma_1}{\pi}(\cos\pi u/\sigma_1, \sin\pi u/\sigma_1)$, giving $\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos(\pi\sigma/\sigma_1)(\cos(\pi ct/\sigma_1), \sin(\pi ct/\sigma_1))$.

• $j_\mu^a = \mathcal{P}_\mu^a$ (where $a = \sigma, \tau$) is the conserved Noether current for spacetime translation invariance, $\delta X^\mu = \epsilon^\mu$. The string equations of motion are equivalent to the worldsheet conservation of this current: $\partial_a j^a_\mu = 0$. The spacetime momentum of the string is the corresponding conserved charge: $p^\mu = \int d\sigma \mathcal{P}^\tau_\sigma$. (More generally, it is $\int (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau)$.) This is conserved for the closed string or open Neumann BCs. Not conserved for Dirichlet BCs.

The Lorentz symmetry comes from the worldsheet symmetry $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$, which is a symmetry if $\epsilon^{\mu\nu} = \epsilon^{[\mu\nu]}$. The assocaited conserved currents are $\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - (\mu \leftrightarrow \nu)$. The corresponding charges $M_{\mu\nu} = \int (\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau)$ are the angular momenta (and M^{0i} is related to the center of mass position at $t = 0$).

• $T_0 \equiv 1/2\pi\alpha'\hbar c$. Consider string in 12 plane. Find that the rotational angular momentum has $M_{12} = \int_0^{\sigma_1} d\sigma (X_1 \mathcal{P}^\tau_2 - X_2 \mathcal{P}^\tau_1)$, which using above $\vec{X}(t,\sigma)$ and $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2} \partial_t \vec{X}$, leads to $M_{12} = \sigma_1^2 T_0/2\pi c$. Since $\sigma_1 = E/T_0$ and $M_{12} = J$, this gives $J = \alpha' \hbar E^2$, which is the Regge trajectory observation of the early '70s. $\ell_s = \hbar c \sqrt{\alpha'}$.

• Aside, for later: the string worldsheet analog of $S_{particle} \supset \int q A_\mu dx^\mu$ is $S_{string} \supset - \int_\Sigma B_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu d\sigma d\tau$.

• Generalize static gauge (to eventually get to light cone gauge). Consider e.g. gauge $n_\mu X^\mu = \lambda \tau$ for time-like n_μ. More generally, can pick

$$n \cdot \mathcal{P}^\sigma = 0, \qquad n \cdot X = \beta \alpha' (n \cdot p) \tau, \qquad n \cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau,$$

where $\beta = 2$ for open strings and $\beta = 1$ for closed strings. These lead to (1) and (2) and (3).