4/24/09 Lecture outline

⋆ Reading: Zwiebach chapter 6, 7, 8.

• Particle $q(\tau)$ vs field $\phi(\xi^\alpha)$ for $\alpha = 0, \ldots d_W - 1$: particle is the case of a single ξ, $d_W = 1$, vs more than one for a field (e.g. $\vec{E}(t, \vec{x})$). Fields have

$$S = \int_\Sigma d^{d_W}\xi \mathcal{L}(\phi, \partial_\alpha \phi),$$

the variation is

$$\delta S = \int_\Sigma d^{d_W}\xi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi}\right)\delta\phi + \int_{\partial\Sigma} \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi}\delta\phi (d^{d_W-1}\xi)^\alpha,$$

where the last term is the boundary contribution, obtained by integrating a total derivative using Gauss' law. The Euler/Lagrange equations are thus

$$\left(\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\right) = 0,$$

where we included an extra index a to be more general.

We also have to ensure that the boundary term vanishes, which is done by requiring either $\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a} n^\alpha|_{\partial\Sigma} = 0$, where n^α is perpendicular to the boundary, or by requiring that ϕ^a is constant along the boundary, or by a combination of these.

While we're at it, let's recall/quote Noether's theorem, relating continuous symmetries of the action to conservation laws. If $\mathcal{L}$ is invariant under some continuous transformation $\phi^a \to \phi^a + \delta\phi^a$, then there is a conserved quantity j^α:

$$\partial_\alpha j^\alpha = 0 \qquad \text{with} \qquad j^\alpha \sim \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\delta\phi^a,$$

where the conservation law follows from $\delta\mathcal{L} = 0$ and the Euler-Lagrange equations. We'll see that spacetime conservation laws, like spacetime momentum and angular momentum conservation, will arise from such conserved currents on the string worldsheet.

• Recall $S = -mc\int ds + \frac{q}{c}\int A_\mu dx^\mu$ for a relativistic point particle, where the first term is the mass times the proper length of the world-line. For a string world-sheet, we need two parameters, ξ^a, $a = 1, 2$. The string trajectory is $x : \Sigma \to M$, where Σ is the 2d world-sheet, with local coordinates ξ^a, and M is the target space, with local coordinates x^μ. The worldsheet area element is $A = \int d^2\xi \sqrt{|h|}$, where h_{ab} is the worldsheet metric, and $|h|$ is its determinant. Suppose that the target space has metric $g_{\mu\nu}$, with space-time length e.g. $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. By writing $dx^\mu = \partial_a x^\mu d\xi^a$, we get

$$ds^2 = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b}d\xi^a d\xi^b, \qquad \text{so} \qquad h_{ab} = g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b},$$

where this h_{ab} is called the induced metric. So the worldsheet area functional is

$$A = \int d^2\xi \sqrt{\det(g_{\mu\nu}\frac{dx^\mu}{d\xi^a}\frac{dx^\nu}{d\xi^b})}.$$

• For strings in Minkowski spacetime, we write it instead as $X^\mu(\tau, \sigma)$. There is also a needed minus sign, as the area element is $\sqrt{|g|}$, actually involves the absolute value of the determinant, and the determinant is negative (just like $\det\eta = -1$). So

$$A = \int d\tau d\sigma \sqrt{(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma})^2 - (\frac{\partial X}{\partial \tau})^2 (\frac{\partial X}{\partial \sigma})^2},$$

where the spacetime indices are contracted with the metric $g_{\mu\nu}$. To get an action with $[S] = ML^2/T$, we have

$$S_{Nambu-Goto} = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

where we define $\dot{X}^\mu \equiv \frac{dx^\mu}{d\tau}$ and $X\mu' \equiv \frac{\partial X^\mu}{\partial \sigma}$ annd T_0 is the string tension, with $[T_0] = [F] = [ML/T^2]$.

The action is reparameterization invariant: can take $(\tau, \sigma) \to (\tau'(\tau, \sigma, \sigma'(\tau\sigma)$ and get $S \to S$. Enormous symmetry/redundancy in choice of (τ, σ); can "fix the gauge" to some convenient choice, and the physics is completely independent of the choice.

• We can write S_{NG} in terms of the Lagrangian density

$$\mathcal{L}_{NG} = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2},$$

and we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}},$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}.$$

The condition $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0.$$

For the open string, $\delta S = 0$ also requires $\int d\tau [\delta X^\mu P^\sigma_\mu]_0^{\sigma_0} = 0$, which requires for each μ index either of the Dirichlet or Neumann BCs, at each end:

$$\text{Dirichlet} \qquad \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \qquad \to \qquad \delta X^\mu(\tau, \sigma_*) = 0,$$

$$\text{Neumann} \qquad \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0.$$

• Static gauge: pick $\tau = t$. Verify sign inside $\sqrt{\cdot}$ in this case.

• In static gauge, there is no KE, so $L = -V$, and verify that string stretched length a has $V = T_0 a$. So $\mu_0 = T_0/c^2$.

• In static gauge, express S in terms of $\vec{v}_\perp = \partial_t \vec{X} - (\partial_t \vec{S} \cdot \partial_s \vec{X})\partial_s \vec{X}$ (with $ds \equiv |d\vec{X}|_{t=const} = |\partial_\sigma \vec{X}||d\sigma|$) to get $L = -T_0 \int ds \sqrt{1 - v_\perp^2/c^2}$. Also get

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2} \frac{(\partial_s \vec{X} \cdot \partial_t \vec{X})\dot{X}^\mu + (c^2 - (\partial_t \vec{X})^2)\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}},$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2} \frac{ds}{d\sigma} \frac{\dot{X}^\mu - (\partial_s \vec{X} \cdot \partial_t \vec{X})\partial_s X^\mu}{\sqrt{1 - v_\perp^2/c^2}}.$$

• Show that endpoints move transversely, $\partial_s \vec{X} \cdot \partial_t \vec{X} = 0$, and at the speed of light, $v = c$, for the free (Neuman) BCs, using fact that $\mathcal{P}^{\sigma\mu} = 0$.

• Choose σ parameterization such that

$$\partial_\sigma \vec{X} \cdot \partial_\tau \vec{X} = 0 \qquad \text{and} \qquad d\sigma = \frac{ds}{\sqrt{1 - v_\perp^2/c^2}} = \frac{dE}{T_0}.$$

The last equation is equivalent to $(\partial_\sigma \vec{X})^2 + c^{-2}(\partial_t \vec{X})^2 = 1$. With this worldsheet gauge choice,

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\partial_t X^\mu = \frac{T^0}{c^2}(c, \vec{v}_\perp), \qquad \mathcal{P}^{\sigma,\mu} = -T_0 \partial_\sigma X^\mu = (0, -T_0 \partial_\sigma \vec{X}).$$

The equation of motion is then simply $(\partial_t^2 - c^2 \partial_\sigma^2)\vec{X} = 0$.

• $j^a_\mu = \mathcal{P}^a_\mu$ (where $a = \sigma, \tau$) is the conserved Noether current for spacetime translation invariance, $\delta X^\mu = \epsilon^\mu$. The string equations of motion are equivalent to the worldsheet conservation of this current: $\partial_a j^a_\mu = 0$. The spacetime momentum of the string is the corresponding conserved charge: $p^\mu = \int d\sigma \mathcal{P}^\tau_\sigma$. (More generally, it is $\int(\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau)$.) This is conserved for the closed string or open Neumann BCs. Not conserved for Dirichlet BCs.

The Lorentz symmetry comes from the worldsheet symmetry $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$, which is a symmetry if $\epsilon^{\mu\nu} = \epsilon^{[\mu\nu]}$. The assocaited conserved currents are $\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - (\mu \leftrightarrow \nu)$. The corresponding charges $M_{\mu\nu} = \int(\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau)$ are the angular momenta (and M^{0i} is related to the center of mass position at $t = 0$).

• $T_0 \equiv 1/2\pi\alpha'\hbar c$. Consider string in 12 plane. Find that the rotational angular momentum has $J = \alpha'\hbar E^2$, which is the Regge trajectory observation of the early '70s. $\ell_s = \hbar c\sqrt{\alpha'}$.

• Aside, for later: the string worldsheet analog of $S_{particle} \supset \int q A_\mu dx^\mu$ is $S_{string} \supset -\int_\Sigma B_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu d\sigma d\tau$.