4/10/09 Lecture outline

⋆ Reading: Zwiebach chapter 3.

• Use units where Maxwell's equations are $\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}$, $\nabla \cdot \vec{B} = 0$, $\nabla \cdot \vec{E} = \rho$, $\nabla \times \vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E}$. The first two equations can be solved by introducing the scalar and vector potential: $\vec{B} = \nabla \times \vec{A}$, $\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\phi$. There is a redundancy here, called invariance under gauge transformation, because the physical quantities $\vec{E}$ and $\vec{B}$ are invariant under

$$\phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t}, \qquad \vec{A} \to \vec{A} + \nabla f, \tag{1}$$

for an arbitrary function $f(t, \vec{x})$. This initially dull sounding invariance takes a fundamental role in modern high energy physics: such local (because f can vary locally over space-time) gauge symmetries are in direct correspondence with forces!

• Maxwell's equations in relativistic form. Like last time, $x^\mu = (ct, \vec{x})$ and also use $\partial_\mu = (c\partial_t, \nabla)$ (and thus $\partial^\mu = (-c\partial_t, \nabla)$). $\vec{E}$ and $\vec{B}$ combine into an antisymmetric, 2-index, 4-tensor $F_{\mu\nu} = -F_{\nu\mu}$, via $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk}B^k$, i.e.

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

As usual, we can raise and lower indices with $\eta_{\mu\nu}$, e.g. $F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}F_{\lambda\sigma}$ and with the book's sign convention this gives a minus sign each time a time component is raised or lowered. So $F^{0i} = -F_{0i}$ and $F^{ij} = F_{ij}$, where i and j refer to the spatial components, i.e. the matrix $F^{\mu\nu}$ is similar to that above, but with $\vec{E} \to -\vec{E}$.

Under Lorentz transformations, $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$, the electric and magnetic fields transform as $F^{\mu'\nu'} = \Lambda^{\mu'}{}_\sigma \Lambda^{\nu'}{}_\rho F^{\sigma\rho}$. Sources combine into a 4-vector as $j^\mu = (c\rho, \vec{j})$, and charge conservation is the Lorentz-invariant equation $\partial_\mu j^\mu = 0$. Maxwell's equations in relativistic form are $\partial_{[\mu}F_{\rho\sigma]} = 0$, and $\partial_\lambda F^{\mu\lambda} = \frac{1}{c}j^\mu$ (this convention, with indices not next to each other contracted, is peculiar to the $(-+++)$ choice of $\eta_{\mu\nu}$), which exhibits that they transform covariantly under Lorentz transformations. The scalar and vector potential combine to the 4-vector $A^\mu = (\phi, \vec{A})$ and the first two Maxwell equations are solved via $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$. The gauge invariance is $A^\mu \to A^\mu + \partial^\mu f$. We can e.g. choose Lorentz gauge, where $\partial_\mu A^\mu = 0$. Physics is independent of choice of gauge, but some are sometimes more convenient than others along the way, depending on what's being done. In Lorentz gauge, the remaining Maxwell equations are $\partial_\mu \partial^\mu A^\nu = -\frac{1}{c}j^\nu$ (still some gauge freedom).

In empty space we set $j^\mu = 0$ and the plane wave solutions are $A^\mu = \epsilon^\mu(p)e^{ip\cdot x}$, where $p^2 = 0$ (massless) and $p \cdot \epsilon = 0$. Can still shift $\epsilon^\mu \to \epsilon^\mu + \alpha p^\mu$, so 2 independent photon polarizations ϵ^μ.

• The action of a relativistic point particle in the presence of electric and magnetic fields is

$$S = \int (-mcds + \frac{q}{c} A_\mu dx^\mu), \tag{2}$$

which is manifestly relativistically invariant. Note also that, under a gauge transformation, we have $S \to S + \frac{qf}{c}$, which does not affect the equations of motion (just as changing the Lagrangian by a total time derivative does not).

The lagrangian is thus $L = -mc\sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c}\vec{v} \cdot \vec{A} - q\phi$. The momentum conjugate to $\vec{r}$ is $\vec{P} = \partial L/\partial \vec{v} = m\vec{v}/\sqrt{1 - \vec{v}^2/c^2} + \frac{q}{c}\vec{A}$. The Hamiltonian is $H = \vec{v} \cdot \vec{P} - L = \sqrt{m^2c^4 + c^2(\vec{P} - \frac{q}{c}\vec{A})^2} + q\phi$.

In the non-relativistic limit we have $H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 + q\phi$, where $\vec{P} - \frac{q}{c}\vec{A} = m\vec{v}$.

The electric and magnetic fields themselves have a lagrangian, with action

$$S = \int d^4x \mathcal{L}, \qquad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{c} A_\mu j^\mu.$$

• In quantum mechanics, we have $[x^k, P^\ell] = i\hbar\delta^{k,\ell}$, so we replace $\vec{P} \to -i\hbar\nabla$ in position space. The S.E. is $H\psi = i\hbar\partial_t\psi$. Note that the derivatives only appear in the "covariant derivative" combination

$$D_\mu = \partial_\mu - i\frac{q}{\hbar c} A_\mu. \tag{3}$$

This is crucial for gauge invariance of physics.

The reason is that gauge invariance is more interesting in quantum theory, as the wavefunction changes under gauge transformations:

$$\psi(t, \vec{x}) \to e^{iqf(t,\vec{x})/\hbar c} \psi(t, \vec{x}), \tag{4}$$

which leaves the probability density and current unchanged. (One way to see this is via $\psi \sim e^{iS/\hbar}$ and noting from the above expression for S that that $S \to S + qf/c$.)

The above covariant derivatives have the property that $D_\mu\psi \to e^{iqf/\hbar c} D_\mu\psi$ under a gauge transformation, with the shift $A_\mu \to A_\mu + \partial_\mu f$ canceling the bad term $\sim \partial_\mu f$. Because derivatives are all covariant, the local parameter $f(x)$ always only enters as an overall phase, which remains physically unobservable upon computing probability $|| \cdot ||^2$.

Gauge invariance says that physics observables can't notice gauge transformations by arbitrary $f(x)$. This phase transformation is called $U(1)$ gauge invariance, i.e. we can take $\psi \to U(x)\psi$, where $U(x) = e^{iqf/\hbar c}$ is an arbitrary local $U(1)$ symmetry transformation. This is why electromagnetism is called a $U(1)$ gauge theory in modern high energy physics, where gauge symmetries are fundamental, and in direct correspondence with the fundamental forces. Each of the 4-known forces is associated with a gauge invariance. (Gravity's is general coordinate invariance.)

The $U(1)_{EM}$ symmetry is the symmetry of rotating a circle. In Kaluza-Klein theory, this circle is that of the compact 5-th dimension! Since charge is quantized, $q = ne$, where $-e$ is the charge of an electron, the gauge symmetry above doesn't even change the wavefunction if $f \to f + 2\pi\hbar c/e$

Another idea for charge quantization: monopoles and Dirac quantization. In vacuo, Maxwell's equations are symmetric under $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$ (in relativistic notation, $F^{\mu\nu} \to \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$). Dirac string and $e^{iS/\hbar} \to e^{iS/\hbar} + e^{ie\oint \vec{A}\cdot d\vec{x}/\hbar c}$ so $\oint \vec{A}\cdot d\vec{x} = \int \vec{B}\cdot d\vec{a} = 4\pi g$ where $eg = \frac{1}{2}\hbar c n$, with n an integer. We haven't seen a monopole yet, but inflation could have removed them. Also in GUTs, $U(1)_{EM}$ is part of a larger symmetry, which leads to monopoles and charge quantization.

• Electromagnetism in other dimensions: $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$ and $\partial_\mu F^{\mu\nu} = \frac{1}{c}j^\nu$. So e.g. a point charge q makes an electric field with $\nabla \cdot \vec{E} = q\delta^d(\vec{x})$ in a world with $D = d+1$ spacetime dimension (the $+1$ is the time dimension, and there are d spatial directions), so $\int_{S^{d-1}} \vec{E} \cdot d\vec{a} = q$. Thus $\vec{E} = E(r)\hat{r}$ with $E(r) = q/r^{d-1}vol(S^{d-1})$, where $vol(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit sphere surrounding the charge. Finally, we get that a point charge makes electric field given by $E(r) = \Gamma(d/2)q/2\pi^{d/2}r^{d-1}$. For $d = 3$, get $E(r) = q/4\pi r^2$, which is the usual answer in these units.

• Gravity has general coordinate invariance, $x^\mu \to x^{\mu'}(x^\mu)$. At the linearized level, take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $x^{\mu'} = x^\mu + \epsilon^\mu(x)$, with $\delta h_{\mu\nu} \approx \partial_{(\mu}\epsilon_{\nu)}$.

Recall $m_P = \sqrt{\hbar c/G} = 2.176 \times 10^{-5}g$.

In 4d, we have gravitational potential given by $V_g^{(4)} = -GM/r$, which solves $\nabla^2 V_g^{(D)} = 4\pi G^{(D)}\rho_m$. This is taken to be the gravitational potential equation in any spacetime dimension, with gravitational force taken to be $F = -m\nabla V_g$. In $\hbar = c = 1$ units, get $G = \ell_P^{D-2}$ in D spacetime dimensions. Get $G^D = GV_C$, where V_C is the compactification volume.

$\ell_C = \ell_P^{(D)}(\ell_P^{(D)}/\ell_P)^{2/(D-4)}$, can imagine e.g. $\ell_P^{(D)} \sim 10^{-18}cm$ instead of $\ell_P \sim 10^{-33}cm$ (i.e. lower gravitational physics effects to $M_P^{(D)} \sim 20TeV$ from $M_P \sim 10^{16}TeV$) which for $D = 6$ gives $\ell_C \sim 10^{-3}cm$ – large extra dimensions.