4/29/08 Lecture 9 outline

• The postulates of quantum mechanics:

1. The state of the system is given by a ket $|\psi(t)\rangle$. (Not directly measurable!)

2. Physical observables (except time) are replaced with Hermitian operators. The possible measured values of any observable A are the various eigenvalues a_i of A.

3. The probability of measuring that observable A has value a_i in the state $|\psi\rangle$ is $|\langle a_i|\psi\rangle|^2$.

4. After measuring that observable A has value a_i, the measurement itself radically affects the system: the wavefunction (immediately) collapses, $|\psi\rangle \to |a_i\rangle$.

5. The time evolution of the system (aside from the measurement effect above) is given by the Schodinger equation: $i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$, where H is the Hamiltonian.

• Fits with what we said last time: expectation values in state $|\psi\rangle$ by $\langle A\rangle \equiv \langle\psi|A|\psi\rangle = \sum_i a_i|\langle a_i|\psi\rangle|^2$. As mentioned, we can interpret the above as saying that a_i are the possible measured values of observable A, and $|\langle a_i|\psi\rangle|^2$ are the probabilities.

• An example: Schrodinger's cat. The observable is whether his cat is alive or dead. The eigenstates are $|\text{alive}\rangle$ and $|\text{dead}\rangle$. Before an observer opens the box, the cat can be in a quantum superposition of being both alive and dead, $|\text{cat}\rangle = c_1|\text{alive}\rangle + c_1|\text{dead}\rangle$, where c_1 and c_2 are complex constants with $|c_1|^2 + |c_2|^2 = 1$. Once the box is opened, there is a probability of $|c_1|^2$ that the observer finds the cat alive (forcing the the wavefunction collapse to $|\text{cat}\rangle = |\text{alive}\rangle$) and a probability of $|c_2|^2$ that the observer finds the cat dead (forcing the wavefunction collapse to $|\text{cat}\rangle = |\text{dead}\rangle$). In a single experiment, the observer will measure either one outcome or the other. (The mixed nature of the state before opening the box can only be noticed indirectly.)

• Another example, with polarization sheets. Two orthogonal sheets don't pass any light. Putting in an extra middle sheet, oriented at angle θ, allows light to pass. Each sheet measures the polarization of each photon, and allows photon to pass if its polarization is along the sheet axis, or not if it is orthogonal. At each stage, each photon is either passed or absorbed, with probabilities determined by decomposing the photon state into the 2d basis of either parallel or orthogonal to the polarization axis.

• We will often need to generalize our vector space concepts from spaces with finite dimension K to infinite dimensional vector spaces. As an example, recall that Fourier's theorem states that any function $\psi(x)$, defined for $0 \leq x \leq a$, which satisfies $\psi(0) = \psi(a) = 0$, can be written as a superposition of the u_n defined above:

$$\psi(x) = \sum_{n=1}^{\infty} c_n u_n(x),$$

where

$$c_n = \int_0^a dx\ u_n(x)^* \psi(x).$$

This can be phrased in Dirac's notation as expanding a vector in an orthonormal basis:

$$|\psi\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\psi\rangle$$

$$\psi(x) = \langle x|\psi\rangle \qquad u_n(x) = \langle x|n\rangle,$$

$$c_n = \langle n|\psi\rangle = \int_0^a dx \langle n|x\rangle\langle x|\psi\rangle.$$

• Replace physical observables, like x, p, E, etc. with Hermitian operators. The observed quantities are the eigenvalues. Write e.g.

$$\widehat{x}|x\rangle = x|x\rangle, \qquad \widehat{p}|p\rangle = p|p\rangle, \qquad H|E\rangle = E|E\rangle.$$

These operators generally act in an infinite dimensional space, the Hilbert space, but this generally doesn't complicate things much (from a physicist's perspective).

• As we have discussed, the operators $\widehat{x}$ and $\widehat{p}$ satisfy $[x,p] = i\hbar$. This is related to the fact that the momentum p generates translations of x. The fact that they don't commute means that they don't have simultaneous eigenvectors, they can't be simultaneously diagonalized.

• For any ket $|\chi\rangle$, we have

$$\langle x|\widehat{x}|\chi\rangle = x\langle x|\chi\rangle \qquad \text{and} \qquad \langle p|\widehat{p}|\chi\rangle = p\langle p|\chi\rangle.$$

$$\langle x|\widehat{p}|\chi\rangle = -i\hbar\frac{d}{dx}\langle x|\chi\rangle \qquad \text{and} \qquad \langle p|\widehat{x}|\chi\rangle = i\hbar\frac{d}{dp}\langle p|\chi\rangle.$$

These relations are consistent with $[\widehat{x}, \widehat{p}] = i\hbar$.

Their separate eigenkets satisfy $\langle x'|x\rangle = \delta(x - x')$, and $\langle p'|p\rangle = \delta(p - p')$. The completeness relations are

$$\int_{-\infty}^{\infty} dx|x\rangle\langle x| = 1_{op} \qquad \int_{-\infty}^{\infty} dp|p\rangle\langle p| = 1_{op}.$$

The relation between these bases is $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$. Note that this satisfies $\langle x|p|p\rangle = (-i\hbar \frac{d}{dx})\langle x|p\rangle = p\langle x|p\rangle$.

- Relate this to what we saw before about Fourier transforms and computing e.g. $\langle f(x)\rangle$ and $\langle F(p)\rangle$ in position or momentum space. The wavefunction in position space is $\psi(x) = \langle x|\psi\rangle$. The wavefunction in momentum space is $\phi(p) = \langle p|\psi\rangle$. The Fourier transform is just a change of basis, using $\langle x|p\rangle$.