4/22/08 Lecture 7 outline

• Last time: we're interested in complex vectors, and then the condition is $\widehat{e}_i \cdot \widehat{e}_j^* = \delta_{ij}$. Then $v_i = \vec{v} \cdot e_i^*$. Consider $\vec{v}$ as a matrix v, with 1 column and K rows. The inner product of two vectors v and w is then $\langle w|v\rangle \equiv w^\dagger v = \sum_{i=1}^K w_i^* v_i$. Note that this is a single, complex number, and that $\langle w|v\rangle^* = \langle v|w\rangle$. It follows that $\langle v|v\rangle$ is real, non-negative, and only vanishes if the vector $v = 0$.

• Dirac's bra-ket notation. The abstract vector is now represented by the ket, $|v\rangle$. We also have bras, like $\langle w| = (|v\rangle)^\dagger$. Multipling a bra and a ket gives the inner product, the braket, like $\langle w|v\rangle$. With this notation, we have basis vectors $|e_i\rangle$, and we have $v_i = \langle e_i|v\rangle$. So we can write $|v\rangle = \sum_i |e_i\rangle\langle e_i|v\rangle$. The basis vectors thus satisfy:

$$\langle e_i|e_j\rangle = \delta_{ij} \qquad \text{(orthonormality)}$$

$$\sum_{i=1}^K |e_i\rangle\langle e_i| = 1_{K\times K} \qquad \text{(completeness).}$$

The RHS of the second relation is a unit matrix. More generally, when we represent vectors as matrices, linear operators acting on them are represented by $K \times K$ matrices, and they simply act by matrix multiplication. Multiplying a bra $\times$ ket as $\langle w|v\rangle$ is like multiplying a row and a column vector, it gives a number, the bracket. But multiplying ket $\times$ bra, like $|v\rangle\langle w|$ gives an operator – multiplying a column vector by a row vector gives a matrix.

In the above notation, we can write a general operator A as

$$A = \sum_{ij=1}^K |e_i\rangle\langle e_i|A|e_j\rangle\langle e_j|,$$

where $\langle e_i|A|e_j\rangle = A_{ij}$ are the matrix elements, and the other pieces are the basis elements for $K \times K$ matrices. Note that we can get the above by using the completeness relation twice.

The adjoint operation acts as

$$A^\dagger = \sum_{ij=1}^K |e_j\rangle\langle e_j|A^\dagger|e_i\rangle\langle e_i|,$$

The order is reversed, and bras and kets are exchanged. Note that $\langle e_j|A^\dagger|e_i\rangle = \langle e_i|A|e_j\rangle^*$.

• The equation for an eigenvector and eigenvalue is $A|a_i\rangle = a_i|a_i\rangle$, where the eigenvector is labeled by the eigenvalue a_i, for $i = 1 \ldots K$.

• Suppose A is Hermitian, $A^\dagger = A$. Then $\langle a_i|A|a_i\rangle^* = a_i^*\langle a_i|a_i\rangle = \langle a_i|A^\dagger|a_i\rangle = a_i\langle a_i|a_i\rangle$, from which it follows that $a_i = a_i^*$; the eigenvalues of Hermitian operators are real.