4/15/08 Lecture 5 outline

• Last time, get 2-slit interference e.g. from $\psi_1 = e^{ipx_1/\hbar}$ and $\psi_2 = e^{ipx_2/\hbar}$, with $p = \hbar = h/\lambda$, get $|\psi_1 + \psi_2|^2 = 4\cos^2(\pi\Delta x/\lambda)$, so constructive interference if $\Delta x \equiv x_1 - x_2 = n\lambda$ and destructive if $x_1 - x_2 = (n + \frac{1}{2})\lambda$, ($n$ is an integer). Now generalize about free particles:

• Free particle of mass m: momentum $\vec{p} = \hbar\vec{k}$ and energy $E = \vec{p}^2/2m = \hbar\omega$. Write wavefunction as a plane wave

$$\psi(\vec{x}, t) \sim e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar}.$$

Note this has $|\psi|^2$ which is independent of position and time: particle's position is completely unknown, special case of Heisenberg uncertainty principle, $\Delta x \Delta p_x \geq \hbar/2$. This is related via $\vec{p} = \hbar\vec{k}$ to a basic property of Fourier transforms, $\Delta x \Delta k \geq \frac{1}{2}$. Get state of position localized in range Δx by a superposition of plane waves, i.e. Fourier transform, with wavenumber range Δk.

• Uncertainty principle: $\Delta x \Delta p_x \geq \hbar/2$. E.g. try to resolve the location of an electron better by using light of smaller wavelength. But such light has bigger momentum, leads to greater uncertainty in electrons momentum.

• Try to check which slit electron went through, and interference patten goes away.

• Let's review Fourier transforms. To avoid rewriting too much, write Fourier transform in notation of momentum p, with $p = \hbar k$. Write first for just 1d.

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\phi(p), \qquad \phi(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x).$$

This is the Fourier transform and its inverse. The probability of finding particle in range from x to $x + dx$ is $P(x)dx = |\psi(x)|^2 dx$. The probability to find the particle having momentum in the range from p to $p + dp$ is $P(p)dp = |\phi(p)|^2 dp$.

Consider e.g. the case where $\psi(x)$ is a Gaussian, centered at the origin

$$\psi(x) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1/2} \exp(-x^2/4\sigma^2) \to |\phi(p)|^2 = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp(-p^2/2\tilde{\sigma}^2),$$

where $\sigma\tilde{\sigma} = \hbar/2$. Note that the above wavefunction examples are properly normalized, to have total probability one:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 = 1, \qquad \int_{-\infty}^{\infty} |\phi(p)|^2 = 1.$$

Find for this example that $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = \sigma^2$ and $\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 = \tilde{\sigma}^2$, so this case saturates the uncertainty principle inequality (it's an equality for the special case of gaussians).

- The time evolution of a free particle is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{i(px-Et)/\hbar}\phi(p), \qquad \phi(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x,t=0)$$

where $E = p^2/2m$. Packet moves with velocity given by the group velocity, $v_g = d\omega/dk$, which for us is now $v_g = dE/dp = p/m$, which is indeed reasonable: the group velocity equals the particle's velocity. The spreading of wave packets is generally given by $d^2\omega/dk^2$, which for us is d^2E/dp^2, which is $1/m$ for a non-relativistic particle.

For example, suppose

$$\psi(x,t=0) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1/2} e^{i\bar{p}x/\hbar}\exp(-x^2/4\sigma^2) \to |\phi(p)|^2 = \frac{1}{\sqrt{2\pi}\tilde{\sigma}}\exp(-(p-\bar{p})^2/2\tilde{\sigma}^2),$$

which is of the Gaussian form, with average momentum $\bar{p}$. As an example, let's take $\bar{p} = 0$, so it's just a Gaussian, with zero average momentum. At time $t = 0$, it's width is σ. Plugging into above expression for time evolution, find that $\psi(x,t)$ again has the Gaussian form, but with $\sigma \to \sqrt{\sigma + it\hbar/2m\sigma}$. Gives a Gaussian probability, with $\sigma(t) = \sqrt{\sigma^2 + (\hbar t/2m\sigma)^2}$. This spreading of the packet is just like diffraction: the more the particle is localized, the more it will later spread out. E.g. an electron, localized initially in $\sigma = 10^{-10}m$ spreads to 1AU ($= 8 \times 60 \times 3 \times 10^8 m$) in $t \approx 9 \times 10^{-8}s$(!). Or 1 gram in $\sigma = 10^{-6}m$ spreads to $2 \times 10^{-6}m$ in $1.6 \times 10^{19}s$(!). Uncertainty principle again.