4/8/08 Lecture 3 outline

• Planck's fix: assume radiation of frequency ω can only be absorbed or emitted in quantized amounts, given by $E = n\hbar\omega$ for integer n. Gives energy density of the glowing light in the cavity:

$$u(\omega, T) = \left(\frac{\hbar\omega}{e^{\hbar\omega/k_N T} - 1}\right)\left(\frac{\omega^2}{\pi^2 c^3}\right).$$

(Planck originally wrote this in terms of $\nu = \omega/2\pi$ and $h = 2\pi\hbar$). For low frequencies, $\hbar\omega \ll k_B T$, the first factor $\to k_B T$, and so $u \to u_{cl}$ in this limit. For high frequencies, the first factor goes to zero as $e^{-\hbar\omega/k_B T}$, avoiding the UV catastrophe.

Knowing the energy density inside the cavity also gives the energy flux through the surface (e.g. if there were a hole). The emitted power per unit area per frequency is related to the above energy density by

$$e(\omega, T) = \frac{1}{4} c u(\omega, T) = \frac{1}{4} c \left(\frac{\hbar\omega}{e^{\hbar\omega/k_N T} - 1}\right)\left(\frac{\omega^2}{\pi^2 c^3}\right).$$

This is the famous blackbody spectrum of radiated power. It appears everywhere in Nature, e.g. the radiated power of a star (or the entire universe) is given by this formula. Fits beautifully all the experimentally observed data, for $h = 2\pi\hbar = 6.6261 \times 10^{-34} J \cdot s$. (Explain the 1/4: light has velocity c, but only component perpendicular to an area element counts. Let the normal to the area element be $\hat{z}$ and use spherical coordinates, with light direction given by θ, ϕ and our desired flux is then $e = cu\langle\cos\theta\rangle$. Here we average $\langle\cos\theta\rangle = \int_{hemi} \cos\theta d\Omega / \int_{hemi} d\Omega$, where $d\Omega = \sin\theta d\theta d\phi$ and *hemi* is because only θ between 0 and $\pi/2$ leads to a flux out of the area element (for θ between $\pi/2$ and π, the flux is inward); this gives $\langle cos\theta\rangle = 1/4$.)

The above expression is power radiated per area per frequency range. Integrating it over all frequency, get the Stephan-Boltzmann result for the total *power per unit area*: $e_{total}(T) = \int_0^\infty e(\omega, T) d\nu = \sigma T^4$, with $\sigma = 2\pi^5 k_B^4 / 15 c^2 h^3$. These relations are very useful in astrophysics and cosmology.

[Aside on the derivation: the thermal average energy of a system with energy levels ϵ_n is $\overline{E} = \sum_n \epsilon_n P(\epsilon_n)$, where $P(\epsilon_n) = e^{-\epsilon_n/kT} / \sum_m e^{-\epsilon_m/kT}$ is the probability of the state having energy ϵ_n. Using $\epsilon_n = n\hbar\omega$, this gives $\overline{E} = -\frac{\partial}{\partial\beta} \ln(\sum_{n=0}^\infty x^n)$ where $\beta \equiv 1/kT$ and $x \equiv e^{-\beta\hbar\omega}$. So $\overline{E} = \frac{\partial}{\partial\beta} \ln(1 - e^{-\beta\hbar\omega}) = \hbar\omega/(e^{\hbar\omega/kT} - 1)$.]

The quantization of energy is more noticeable for larger frequencies. A natural unit of energy for atoms is the eV, $1eV = 1.602 \times 10^{-19} J$. In these units, $h \approx 4.13567 \times$

$10^{-15} eV/Hz$. As an example, optical frequencies are order few times $10^{14} Hz$, and the quanta of energy are of order $2eV$. (Visible light has $\lambda \sim 4 - 7 \times 10^{-7} m$.)

In 1905, Einstein suggests that light is made up of particles, "photons" of energy $E = \hbar\omega$ and momentum $\vec{p} = \hbar\vec{k}$ (note that $\omega = c|\vec{k}|$ then gives $E = c|\vec{p}|$, which is the $m = 0$ case of $E = \sqrt{(c\vec{p})^2 + (mc^2)^2}$; so the photon is a massless particle). These quanta also fits with a number of other experiments of the early 1900s, whose results could not be explained by the classical description of light as a wave.

• Photoelectric effect. Shine light on metal. Electrons kicked out. Measure maximum K.E. K_{max} via stopping voltage, $eV_s = K_{max}$. Find K_{max} doesn't depend on the brightness of the light, and there is no time lag. The intensity of the light only affects the number of ejected electrons. Not what classical physics would give. Instead, find K_{max} depends linearly on frequency ω. Suggests light as quanta of energy, photons, of energy $E = h\nu = \hbar\omega$. The ejected electron has kinetic energy $K_{max} = \hbar\omega - W$, where W is the work function, depends on the metal. Plot K_{max} vs ω, slope gives $\hbar$, agrees with Plank's. It works. In practice, $W \sim \mathcal{O}(1) eV$, with $1eV = 1.602 \times 10^{-19} J$.

• Compton effect (1924). Shine X-rays through a thin metal foil, and measure intensity of scattered light as a function of λ and the scattering angle θ. Classical wave prediction: scattered intensity has $I_{wave}(\lambda, \theta) = I_0 \cos^2\theta$, with the same frequency as the incoming light. But instead the experiment fits with the picture that photons scatter off electrons just like scattering (relativistic) particles. Photons carry momenta $\vec{p} = \hbar\vec{k}$. Fits with $E = \hbar\omega$: then $E = pc$, since light travels with speed c, it must be massless. Scatter photons off electrons. Energy momentum conservation, $p_1 + p_2 = p_3 + p_4$. These are 4-vectors. We can write $p^\mu = (E/c, \vec{p})$ and recall that $p^2 = (E/c)^2 - \vec{p}^2 = (mc)^2$ in any frame of reference. Take $p_1 = (p, p, 0, 0)$, $p_2 = (m_e c, 0, 0, 0)$, $p_3 = (p', p'\cos\theta, p'\sin\theta, 0)$, and p_4 is the final momentum of the electron. Write $p_1 + p_2 - p_3 = p_4$ and square to get finally

$$\lambda' - \lambda = \lambda_c(1 - \cos\theta) \qquad \lambda_c = \frac{h}{m_e c} = 2.426 \times 10^{-12} m.$$

Change in wavelength is independent of intensity and time of exposure, depends only on scattering angle.

• Finish reviewing history. Balmer formula for hydrogen spectral lines, $\lambda^{-1} = R(n_f^{-2} - n_i^{-2})$, $R \approx 1.01 \times 10^{-7} m^{-1}$. Rutherford scattering suggests hydrogen structure: proton in center, with orbiting electron. Classically would radiate and spiral in, would get radiation of frequency given by $e^2/4\pi\epsilon_0 r^2 = m_e \omega_e^2 r$, so $\omega_e^2 = (e^2/4\pi\epsilon_0 m_e) r^{-3}$ would get bigger as electron spirals in. Bohr (1913): $\omega_\gamma \neq \omega_e$. Instead quantized energy levels, and $E_\gamma = E_i - E_f$. Get Balmer formula if $E_n = -2\pi\hbar c R/n^2$. Bohr argues for this by postulating that $L = m_e v r = m_e \omega r^2$ is quantized, $L = n\hbar$. Then $E = \frac{1}{2}mv^2 - e^2/4\pi\epsilon_0 r = -e^2/8\pi\epsilon_0 r = -\frac{1}{2}m_e(e^2/4\pi\epsilon_0)^2 L^{-2}$ agrees with Balmer. Also can write $L^2 = m_e e^2 r/4\pi\epsilon_0$ to get an expression for the size of the atom, $r_n = n^2 a_0$, where a_0 is the groundstate radius, $a_0 = \hbar c/m_e c^2\alpha$, where $\alpha \equiv e^2/4\pi\epsilon_0\hbar c \approx 1/137$, gives $a_0 \approx 0.529 \times 10^{-10} m$ in agreement with observation.

Generalization to Bohr-Sommerfeld quantization: $\oint p dx = nh$.

Correspondence principle: $\lim_{n\to\infty}$ of quantum results should agree with classical physics. Classical result is found from $e^2/4\pi\epsilon_0 r^2 = m_e \omega_e^2 r$ and $L = m_e \omega r^2$ so $L^3 = m_e^3 \omega_e^{-1}(e^2/4\pi\epsilon_0 m_e)^2$. Writing $L = n\hbar$, this gives $\omega_e = m_e c^2\alpha^2/\hbar n^3$. On the other hand, $E_i \to E_f$ transition gives a photon of frequency $\omega_\gamma = (E_i - E_f)/\hbar = -(2\hbar)^{-1} m_e c^2 \alpha^2 (n_i^{-2} - n_f^{-2})$. Take $n_i = n$ and $n_f = n - 1$, get $\omega_\gamma = m_e c^2 \alpha^2 (2n-1)/2\hbar n^2 (n-1)^2$, which agrees with ω_e for $n \to \infty$.