5/27/08 Lecture 16 outline

• Last time; $H_{SHO} = p^2/2m + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + \frac{1}{2})$, where $a = \sqrt{\frac{m\omega}{2\hbar}}\widehat{x} + i(2m\omega\hbar)^{-1/2}\widehat{p}$ and $a^\dagger$ satisfy $[a, a^\dagger] = 1$. We saw $a^\dagger a|\lambda\rangle = \lambda|\lambda\rangle$, with $\lambda \geq 0$ and $a|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle$, and $a^\dagger|\lambda\rangle = \sqrt{\lambda+1}|\lambda+1\rangle$. Note that $a^k|\lambda\rangle \sim |\lambda - k\rangle$.

• Must then have a state annihilated by a, i.e. $\lambda = n$ and $a|0\rangle = 0$.

• Thus $H_{sho}|n\rangle = E_n|n\rangle$ with $E_n = (n + \frac{1}{2})\hbar\omega$. Moreover, $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$.

• Using $a = \sqrt{\frac{m\omega}{2\hbar}}\widehat{x} + i(2m\omega\hbar)^{-1/2}\widehat{p}$, convert the above to position space. E.g. get $\psi_0(x) = (m\omega/\pi\hbar)^{1/4}\exp(-m\omega x^2/2\hbar)$ from $\langle x|a|0\rangle = 0$, as a simple differential equation in position space. Can similarly use a and $a^\dagger$ in position space to get all the $\psi_n(x)$. Defining $y \equiv \sqrt{m\omega/\hbar}x$, get

$$\psi_n(x) = (2^n n!)^{-1/2}(m\omega/\pi\hbar)^{1/4}(y - \frac{d}{dy})^n e^{-y^2/2}.$$

But it's almost always better to work with the basis independent bras, kets, and the operators a and $a^\dagger$.

• The states $|n\rangle$, with $n = 0, 1, 2\ldots$, form an orthonormal, complete basis:

$$\langle n|m\rangle = \delta_{nm} \qquad \text{and} \qquad \sum_{n=0}^{\infty}|n\rangle\langle n| = 1_{op}.$$

The annihilation and operators act on them as

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

To compute things like $\langle n|\widehat{x}^k|m\rangle$ etc, it is useful to express $\widehat{x}$ and $\widehat{p}$ in terms of a and $a^\dagger$, using

$$\widehat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \qquad \widehat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right).$$

• Example: Evaluate Δx and Δp in the state $|1\rangle$. Use $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$ and $\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2$. Note that $\langle 1|a^k|1\rangle = 0$ for all $k > 0$. Thus $\langle x\rangle = \langle p\rangle = 0$ in the state $|1\rangle$. To compute $\langle x^2\rangle$, use $\widehat{x}^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2$, and note that only the 2nd and 3rd terms in $(a + a^\dagger)^2 = a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}$ give a non-zero contribution when sandwiched between $\langle 1|$ and $|1\rangle$. To compute these, use the above expressions for how a and $a^\dagger$ act on $|n\rangle$ to get $a^\dagger a|1\rangle = |1\rangle$, and $aa^\dagger|1\rangle = 2|1\rangle$, so $\langle x^2\rangle = \frac{\hbar}{2m\omega}(1+2)$. Likewise, get $\langle p^2\rangle = -\frac{m\omega\hbar}{2}(-1-2)$. Note that $\Delta x\Delta p = \frac{3}{2}\hbar$, so the uncertainty principle inequality is comfortably satisfied.

• Consider an operator $\mathcal{O}$, which is built up out of the position and momentum operators, without explicit dependence on t. It follows from the quantization of the classical Poisson brackets, or from the Schrodinger equation, that

$$i\hbar \frac{d}{dt}\langle \mathcal{O} \rangle = \langle [\mathcal{O}, H] \rangle.$$

Operators for which $[\mathcal{O}, H] = 0$ can be measured simultaneously with the energy, and their expectation values are conserved in time. This is the quantum analog of classical statements about conserved quantities, which are generally associated with symmetries.

• Our next topic is QM in 3 space dimensions. An important case is when there is rotation symmetry. Then angular momentum is conserved. We can see that because $[H, \vec{L}] = 0$. So we can simultaneously measure energy and some components of angular momentum.

• But note that different components of the angular momentum don't commute! Angular momentum commutation relations $[L_x, L_y] = i\hbar L_z$ and cyclic permutations. Implies that we can't measure these different components simultaneously.