5/22/08 Lecture 15 outline

• Next topic: the harmonic oscillator (SHO), $H = p^2/2m + \frac{1}{2}m\omega^2 x^2$. Appears everywhere in nature, since expanding any potential around a stable equilibrium looks like a SHO to leading order in small fluctuations. Note $\langle H\rangle = |\widehat{p}|\psi\rangle|^2/2m + m\omega^2|\widehat{x}|\psi\rangle|^2/2 \geq 0$, so all eigenvalues of H must be non-negative. In fact, they must be positive because the only way to get zero would be if both terms above vanish, i.e. if there is a state which is a zero eigenstate of both $\widehat{x}$ and $\widehat{p}$. But that is impossible, since $[\widehat{x}, \widehat{p}] = i\hbar$.

• Use the uncertainty principle to estimate the groundstate energy. In this example, we will compute E_0, but in more complicated situations it is useful to know how to approximate the answer using the uncertainty principle. $\langle H\rangle \sim (\Delta p)^2/2m + \frac{1}{2}m\omega^2(\Delta x)^2 \sim \hbar^2/2m(\Delta x)^2 + \frac{1}{2}m\omega^2(\Delta x)^2$, and minimize w.r.t. Δx to find the minimum of $\langle H\rangle$, which is the groundstate energy E_0.

• The differential equation $H\psi_E(x) = E\psi_E(x)$ in position space can be solved, as

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/2} e^{-m\omega x^2/2\hbar} H_n[(\frac{m\omega}{\hbar})^{1/2}x]$$

where the H_n are the Hermite polynomials, $H_0 = 1$, $H_1[y] = 2y$, and $H_{n+1}[y] = 2yH_n[y] - 2nH_{n-1}[y]$. The energy eigenvalues are $E_n = (n + \frac{1}{2})\hbar\omega$.

• The harmonic oscillator, using $a = \sqrt{\frac{m\omega}{2\hbar}}\widehat{x} + i(2m\omega\hbar)^{-1/2}\widehat{p}$ and $a^\dagger$, with $[a, a^\dagger] = 1$ and $H_{sho} = \hbar\omega(a^\dagger a + \frac{1}{2})$.

• Show $a^\dagger a|\lambda\rangle = \lambda|\lambda\rangle$, with $\lambda \geq 0$.

• Show $a|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle$, and $a^\dagger|\lambda\rangle = \sqrt{\lambda + 1}|\lambda + 1\rangle$. Note that $a^k|\lambda\rangle \sim |\lambda - k\rangle$.

• Must then have a state annihilated by a, i.e. $\lambda = n$ and $a|0\rangle = 0$.

• Thus $H_{sho}|n\rangle = E_n|n\rangle$ with $E_n = (n + \frac{1}{2})\hbar\omega$. Moreover, $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$.

• Using $a = \sqrt{\frac{m\omega}{2\hbar}}\widehat{x} + i(2m\omega\hbar)^{-1/2}\widehat{p}$, convert the above to position space. E.g. get $\psi_0(x) = (m\omega/\pi\hbar)^{1/4}\exp(-m\omega x^2/2\hbar)$ from $\langle x|a|0\rangle = 0$, as a simple differential equation in position space. Can similarly use a and $a^\dagger$ in position space to get all the $\psi_n(x)$. Defining $y \equiv \sqrt{m\omega/\hbar}x$, get

$$\psi_n(x) = (2^n n!)^{-1/2}(m\omega/\pi\hbar)^{1/4}(y - \frac{d}{dy})^n e^{-y^2/2}.$$

But it's almost always better to work with the basis independent bras, kets, and the operators a and $a^\dagger$.

• The states $|n\rangle$, with $n = 0, 1, 2 \ldots$, form an orthonormal, complete basis:

$$\langle n|m\rangle = \delta_{nm} \qquad \text{and} \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1_{op}.$$

The annihilation and operators act on them as

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

To compute things like $\langle n|\widehat{x}^k|m\rangle$ etc, it is useful to express $\widehat{x}$ and $\widehat{p}$ in terms of a and $a^\dagger$, using

$$\widehat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \qquad \widehat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right).$$

• Example: Evaluate Δx and Δp in the state $|1\rangle$. Use $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$ and $\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2$. Note that $\langle 1|a^k|1\rangle = 0$ for all $k > 0$. Thus $\langle x\rangle = \langle p\rangle = 0$ in the state $|1\rangle$. To compute $\langle x^2\rangle$, use $\widehat{x}^2 = \frac{\hbar}{2m\omega}(a+a^\dagger)^2$, and note that only the 2nd and 3rd terms in $(a+a^\dagger)^2 = a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}$ give a non-zero contribution when sandwiched between $\langle 1|$ and $|1\rangle$. To compute these, use the above expressions for how a and $a^\dagger$ act on $|n\rangle$ to get $a^\dagger a|1\rangle = |1\rangle$, and $aa^\dagger|1\rangle = 2|1\rangle$, so $\langle x^2\rangle = \frac{\hbar}{2m\omega}(1+2)$. Likewise, get $\langle p^2\rangle = -\frac{m\omega\hbar}{2}(-1-2)$. Note that $\Delta x\Delta p = \frac{3}{2}\hbar$, so the uncertainty principle inequality is comfortably satisfied.

• Explain the names, creation and annihilation operators – and phonons.