5/20/08 Lecture 14 outline

- Recall from last time

$$\frac{\partial}{\partial t}\rho(\vec{x},t) + \nabla\cdot\vec{j} = 0, \qquad \rho \equiv \psi^*\psi, \quad \vec{j} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*).$$

Today we'll continue to consider non-bound particles, and scattering off step potentials. The reflection and transmission coefficients give the probability of the incident wave being reflected and transmitted, respectively,

$$R = J_R/J_I, \qquad T = J_T/J_I,$$

and flux conservation ensures that $R + T = 1$.

- Simplest example of scattering from a step potential,

$$V(x) = V_0\theta(x), \qquad \theta(x) \equiv \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

We will solve the eigenvalue equation $H\psi = E\psi$. Suppose that there is an incoming flux from the left, with energy E. The wavefunction is then of the form

$$\psi_1(x) = e^{ik_1x} + Ae^{-ik_1x},$$

where 1, is for the $x \le 0$ region. k_1 is given by $\hbar k_1 = \sqrt{2mE}$. In region 2, which is $x \ge 0$, we have

$$\psi_2(x) = Be^{ik_2x}$$

where $\hbar k_2 = \sqrt{2m(E - V_0)}$. We chose the solution so that the wave only moves to the right in region 2, because we take the particle to be incoming from $x = -\infty$.

The A term is the reflected part of the wave, ψ_R, and the B term is the transmitted part of the wave, ψ_T.

We solve for A and B by noting that the wavefunction must be continuous. Moreover, for a smooth potential, the derivative of the wavefunction must also be continuous. So

$$1 + A = B \qquad ik_1(1 - A) = -k_2B$$

gives

$$A = \frac{k_1 - k_2}{k_1 + k_2} \qquad B = \frac{2k_1}{k_1 + k_2}$$

The flux in region 1 is

$$J = \frac{\hbar}{2im}(\psi^* \psi' - \psi^{*\prime} \psi) = \frac{\hbar k_1}{m}(1 - |A|^2)$$

The flux in region 2 is

$$J = \frac{\hbar k_2}{m}|B|^2$$

Where

$$\frac{\hbar k_1}{m}|A|^2 = \frac{\hbar k_1}{m}\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \qquad \frac{\hbar k_2}{m}|B|^2 = \frac{\hbar k_1}{m}\frac{4k_1 k_2}{(k_1 + k_2)^2}.$$

The reflection and transmission coefficients are

$$R = J_R/J_I = |A|^2, \qquad T = J_T/J_I = \frac{k_2}{k_1}|B|^2,$$

where flux conservation ensures that $R + T = 1$.

If $E < V_0$, then instead get $\psi_2(x) = Be^{-\kappa_2 x}$, where $\hbar\kappa_2 = \sqrt{2m(V_0 - E)}$. In that case, $R = 1$. Find also $B = 2k_1/(k_1 + i\kappa_2)$.

• Comments on delta function potential, and how the ψ' matching is affected (useful for the HW): integrate the S.E. across the delta function potential to get

$$-\frac{\hbar^2}{2m}\frac{d\psi}{dx}\Big|_{x-\epsilon}^{x+\epsilon} + \int_{x-\epsilon}^{x+\epsilon} V(x)\psi(x) = 0,$$

where the second term only contributes if $V(x)$ has a delta function. Then the above equation shows that ψ' has a specific discontinunity across that x. E.g. if $V(x) = -aV_0\delta(x)$ we get $-(\hbar^2/2m)(d\psi/dx)|_{-\epsilon}^{\epsilon} = aV_0\psi(0)$.

• Step well $V(x) = -V_0(\theta(x - a) - \theta(x + a))$. For $E > 0$, oscillating solutions in 3 regions, work out matching of coefficients by matching $u(x)'/u(x)$ across the boundaries. Find that there is no reflection when the distance $4a$ is an integer number of wavelengths: destructive interference of waves reflected at two edges (again, similar to optics).

• Potential barrier: $V(x) = V_0(\theta(x - a) - \theta(x + a))$. Consider $E < V_0$. Solution for $x < -a$ is $u = e^{ikx} + Ae^{-ikx}$, for $|x| < a$ is $u = Ce^{-\kappa x} + De^{+\kappa x}$ and for $x > a$ is Be^{ikx}. Here $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(V_0 - E)}/\hbar$. Matching gives

$$T = |B|^2 = \frac{(2k\kappa)^2}{(k^2 + \kappa^2)^2 \sinh^2 2\kappa a + (2k\kappa)^2}.$$

This illustrates tunneling: $T \neq 0$ even though $E < V_0$! Can plot T as a function of E/V_0. In the limit where the tunneling is very small, i.e. $\kappa a \gg 1$, get

$$T \approx \frac{16E(V_0 - E)}{V_0^2} e^{-4\kappa a}.$$

This exponential is characteristic of tunneling, and similar to the skin-depth effect in time dependent electric and magnetic fields in a conductor. More generally, $T \sim |B|^2 \sim \exp(-2\hbar^{-1} \int_{barrier} \sqrt{2m(V(x) - E)})$.

Example: decay of heavy elements by α tunneling in nucleus. Model the potential as

$$V(r) = \begin{cases} 0 & \text{if } r < r_0 \\ \frac{Z_1 Z_2}{r} & \text{if } r > r_0 \end{cases}$$

where r_0 is the radius of the nucleus, and $Z_1 = 2$ is the charge of the alpha particle (2 protons and 2 neutrons) and $Z_2 = Z - 2$ is the remaining charge of the nucleus. The energy E of the emitted particles is such that $E < Z_1 Z_2 / r_0$. For example, $Z_1 = 2$, $Z_2 - 2 \approx 90$, and $r_0 \approx 10^{-15} m$ gives barrier height $Z_1 Z_2 / r_0 \approx 25$ MeV. But the emitted alpha particles only have E of the order of 5 to 10 MeV. And yet they escape – thanks to tunneling.