5/13/08 Lecture 12 outline

• Parity symmetry in particle in box example. Shift $x = x_{before} - L/2$, to make manifest the parity $P : x \to -x$ symmetry of $V(x)$. Parity also takes $P : p \to -p$ to preserve $[x, p] = i\hbar$. Parity is an operator, which commutes with $H = p^2/2m + V(x)$ whenever $V(x) = V(-x)$. Since $[H, P] = 0$, they have simultaneous eigenstates, so the energy eigenstates are also parity eigenstates. Indeed,

$$\psi_{E_n}(x) = \begin{cases} \sqrt{\frac{2}{L}}\cos(\frac{n\pi x}{L}) & n = 1, 3, 5, 7\dots \\ \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L}) & n = 2, 4, 6\dots \end{cases},$$

inside the box, and $\psi_{E_n}(x) = 0$ outside the box, and these are indeed parity eigenstates: $P\psi_{E_n}(x) = \psi_{E_n}(-x) = (-1)^{n+1}\psi_{E_n}(x)$.

• The groundstate is the solution with the fewest wiggles, which is always the even parity solution in problems with a parity symmetry. More precisely, the general wavefunction has nodes at x_{node} such that $\psi(x_{node}) = 0$. The more nodes, the smaller the wavelength, and thus the larger the frequency and hence the energy. The groundstate is the wavefunction with the fewest nodes. The odd solutions always have a node at $x = 0$, so they always have at least one extra node as compared with the groundstate.

• Particle in a finite depth box,

$$V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & |x| \geq a \end{cases}.$$

Solve the problem in regions I (left), II (inside), and III (right). There are two classes of solutions: bound states, and unbound. The bound states have $E < 0$, and these energy levels are quantized. The unbound states have $E > 0$, and these energy levels aren't quantized. This is a general aspect of quantum mechancs. It is because states with $E > V(x \to \pm\infty)$ can escape to $x \to \pm\infty$, whereas those with $E < V(x \to \pm\infty)$, cannot, and such bound states necessarily have Schrodinger equation solutions satisfying $\psi_E(x \to \pm\infty) \to 0$.

Let's first consider the bound state solutions for this example. In the classically unallowed regions, there are exponentially decaying solutions of the SE. In the classically allowed region, there are oscillatory solutions. The solution $\psi_E(x)$ must be continuous, and $\psi_E(x)'$ must also be continuous. It is sufficient to match ψ'_E/ψ_E across the boundaries.

Because $[H, P] = 0$, the energy eigenstates are also parity eigenstates. The solutions are then either even:

$$\psi_E(x) = \begin{cases} Ae^{\kappa x} & x < -a \\ B \cos kx & |x| \leq -a \\ Ce^{-\kappa x} & x > a, \end{cases}$$

where $\kappa = \sqrt{2mE}/\hbar$ and $k = \sqrt{2m(V_0 - |E|)}/\hbar$, with $C = A$, or similarly for the odd solution but with cos replaced with sin and $C = -A$.